Advice on Proofs

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1 Introduction

This article offers advice on two important features of mathematics which students often find difficult: proofs and problem-solving. It is aimed primarily at Year 1 Maths students, but of course may be helpful to other students as well. I would have found a document like this very helpful when I was a first year student. I hope the advice I have learnt from three years as a maths undergraduate offers the help I would have liked.

Although the wonderful staff at Edinburgh University will probably teach you many of these things, it may also be helpful to hear it from the perspective of a student.

Naturally, I learn things in a very detailed manner and in such a way that I can teach and explain them. For these reason, I think I am well-suited to offering advice.

2 Introduction to Proofs

Before offering advice on proofs, it makes sense to provide a some additional background on what a proof is, why we need proofs, and what a proof looks like. This should build on what you learnt in school about proofs and help you transition to university-level proofs. However, it is not a comprehensive summary, so is not the place to first learn about proofs.

2.1 What is a proof?

A proof is a logical argument that determines the validity of a claim: a claim is either true or false. A proof starts with a series of assumptions and derives a conclusion through a series of logical steps. A proof has a direction - from assumptions to conclusions. The steps in a proof follow *deductive logic,* which basically means we use 'implies' to deduce each step in the argument. In maths, a statement P *implies*[1] a statement Q (or $P \implies Q$) means that if P is true, then it *must* be that Q is true. For example, if $x = 2$, then it must be true that $x^2 = 4$, so $x = 2$ *implies* $x^2 = 4$. Contrast this with 'implies' often used in everyday life: 'It is sunny, which implies it is warm'. Here, the assumption of 'sunny' is not enough to conclude *definitively* that it will be warm (you get sunny but cold days). The point is implies is used more vaguely in real life, whereas in maths it follows the precise rules of implication.

In another sense a proof is to a mathematician as experimental evidence is to a scientist. Proofs and experiments both - albeit in different ways - tell us what is true and what is not. Though in a sense similar, evidence in maths is different to scientific evidence: in maths, something is not true unless proved so; meaning no amount of evidence (e.g. 'it holds for the first $1,000,000$ terms') is enough to conclude a statement is true or not (unless we check every case). This is why the Riemann Hypothesis and the Goldbach Conjecture remain unsolved, despite the millions of cases providing evidence 'in favour' of the claims.

2.2 Why do we need proofs?

Proofs are essential in mathematics and serve many purposes; but I believe there are three purposes of proofs, which are most appropriate to this article. These are: to convince yourself, to convince others, and to establish mathematical *truth.*

[1] You may argue proofs that show equivalence of statements P, Q use 'equivalence' rather than 'implies' for deduction. However, it is possible to prove that $P \iff Q$ using only 'implies' by proving that $P \implies Q$ and $Q \implies P$. So perhaps this is just a question of style.

Mathematics is all about ideas, so being able to check if an idea is correct - in other words convince yourself, then that is very useful. To check if any idea is correct you can try to prove it. If you are successful in writing a proof which to you seems correct, then you have convinced yourself of your idea. On the other hand, you might find your idea causes contradictions - inconsistencies - with other established theory; in which case, your idea is not correct (but may be upon adjustment).

You are convinced in your idea, so now you must convince others. If your proof is accepted among mathematicians[2] as being correct, then your idea is considered to be true. This is the second strength of mathematical proof. Others may see steps in your reasoning that need further justification or correction, and if eventually you can as it were iron out all of the creases, then you will have convinced other mathematicians of your idea.

Once others are convinced, you have established a true mathematical result. This result is now an established part of mathematical knowledge, against which other ideas will be checked and on which further theory may be built. This result is now *eternal*. All of these things because of proofs. Quite a lot, really!

2.3 What does a proof look like?

A proof is typically written using a combination of natural language (used in ordinary writing) and mathematical symbols (such as $\implies$, $=$), which are used to make writing shorter and easier to understand. A proof has a direction, and it should be readable in the same way prose is, using sentences, punctuation, etc.

Below are two examples of proofs. The first proof is a direct proof, whereas the second proof is an indirect proof (a proof by contradiction). In each case, focus on the way the argument flows - see if you can identify every deduction. Try to understand why each implication is true, and check that all the characteristics of a proof given above are present.

(Note that we don't always write, 'P implies Q'; we may write, 'If P, then Q', or 'Q if P. Also note that 'hence', 'therefore', 'thus', and 'it follows that' all mean the same as 'implies'.)

Claim 1: The square of an odd number is odd.

Proof. Suppose n is an odd number, which means $n = 2k + 1$ for some $k \in \mathbb{Z}$. Then,

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1. \end{aligned}$$

Since $2k^2 + 2k$ is an integer, n^2 is therefore odd, as required.[3]

□

Claim 2: $\sqrt{2}$ is irrational.

Proof. We prove the claim by contradiction; so suppose the contrary, that $\sqrt{2}$ is rational.

This means that $\sqrt{2} = \frac{a}{b}$, for $a, b \in \mathbb{Z}$, with $b \neq 0$ and $\gcd(a, b) = 1$.

From this we have $a = \sqrt{2}b$, which implies $a^2 = (\sqrt{2})^2 b^2 = 2b^2$.

This implies that a^2 is divisible by 2, and is therefore even.

It follows that a must also be even (if a were odd, then a^2 is odd by Claim 1, which contradicts a^2 being even). Hence, $a = 2k$ for some $k \in \mathbb{Z}$.

Then, $2b^2 = a^2 = (2k)^2 = 4k^2 \implies b^2 = 2k^2$.

[2] exactly what is considered acceptable is at times a contentious issue; though, I expect not at an undergraduate level. For an idea of disagreements that have occurred, see this brief summary of disagreement with the proof of Poincaré Conjecture: `https://en.wikipedia.org/wiki/Grigori_Perelman#Current_viewpoints`.

[3] The little square at the end of a proof means 'end of proof', or QED.

Therefore, b^2 is also even, and hence b is even by the same reason as before.

Therefore, $\gcd(a, b) = 2$, which contradicts the assumption that $\gcd(a, b) = 1$.

Thus, our assumption that $\sqrt{2}$ is rational cannot be true, which means that $\sqrt{2}$ must be irrational, as required. $\square$

3 Advice on Proofs

What follows is mostly advice on the mechanics and realities of 'finding', writing, and generally working with proofs, and advice about problem-solving. It is designed to help you improve at *proving* and problem-solving.

Proofs and problem-solving are not unrelated: the *convincing yourself* phase of a proof (see 2.2) is similar to problem-solving. It involves trying to understand how something works, and why, and solving the problem of showing that it is true (or false). On the other hand, the *convincing others* phase of a proof (see, again 2.2) is more closely related to mathematical writing and explanation, since by then you have - you hope - solved the problem, so it remains simply to explain it. You will see what I mean through experience.

The skills involved in proofs and problem-solving are also useful for learning about maths on the whole and for mathematical writing. The process of understanding required to write a proof is similar to the process of understanding involved in deeply learning about a mathematical idea; the process writing a proof is similar to the process of explaining and idea in mathematical writing. Thus, by getting good at proving and problem-solving, you are benefiting your ability to perform other mathematical skills as well.

Before you read on, I must stress that as you get used to proofs you *must not worry.* Although it may feel uncomfortable, or perhaps daunting, at first, they become more familiar and comfortable with time. Through quality practise you *will* get better at them. I can assure you of this.

3.1 Tips for getting started

It is likely that you feel at a loss for how to even start a problem, let alone find a proof for it. The following tips should get you over this first hurdle. They highlight techniques that are fundamental in maths and will apply throughout the entirety of your maths career, so get used to them!

- *Understand what the words mean.* It is absolutely crucial to know the definitions of each word in a claim. For example, to understand the claim,

  'The span of the standard basis of $\mathbb{R}^3$ is a linear subspace of $\mathbb{R}^3$.'

  we must understand what *span, standard basis,* and *linear subspace* mean, and what $\mathbb{R}^3$ is.

- *Try examples.* By considering basic examples of the objects you are working with, you may be able to notice which of their properties are most pertinent to the problem. At the very least, it familiarises you with the objects in question.

- *Understand what the problem is saying.* This requires you to understand the definition of every term, but also to understand how the terms fit together - combined, what are they *saying.* Can you create a mental picture of it? Can you think of a simple example?

  It also requires you to know what exactly you can assume and what you are trying to show. Many times in first year did I assume things I couldn't, or assumed the wrong things, or proved the wrong thing; so be careful!

  If you are not sure what a problem is saying, then ask a friend or a lecturer, as soon as possible. Otherwise, you could be trying to answer a different problem. Meanwhile, attempt a question which you fully understand.

- *Think about what strategy(s) you could use to tackle the problem.* Immediately you have several choices: you could try to prove the claim directly, or by counterexample, or contrapositively, or indirectly (by contradiction), or inductively. Think about which option is most applicable.

- *Understand and be aware of the course notes.* You will be surprised at how many times a problem you are stuck on becomes easier once you find something relevant and helpful in the notes. It helps to read the notes for a course slowly and in full, and is good practice overall.

  If you don't understand a section in the notes, I would suggest noting down exactly what you don't understand, then get help with it or come back to it with more context and see if you can understand it better.

- *Work forwards.* By this I mean, always think what the implications of your assumptions are, and in turn the implications of those. Try to always think, 'what does this tell me?'. Commonly, students spend too much time thinking about what the end goal is. You need to bear the end goal in mind, but you will never get there unless you can draw conclusions, one step at a time.

  Sometimes, however, you may get stuck 'working forwards'. In which case, you may want to try 'working backwards' from the desired conclusion, which means thinking, 'what must precede this?'. You may be able to meet in the middle.

Additionally, you should *put your phone away.* Phones can be a great hindrance to good learning and are immensely distracting. I would suggest turning your phone off and putting it in your bag, as it's incredibly tempting to do the easy thing and look at it, rather than do the hard thing and solve a problem. If you are constantly distracted by your phone, accept that the quality of your learning and work will suffer. Good quality learning and work require focus.

3.2 What do to if I am stuck?

If you are stuck, then don't despair and certainly don't give up. Being stuck is in many ways a good thing, as it is an opportunity to improve and shows that you are going to have to engage your brain to overcome the challenge. In fact, being stuck is probably the most common thing you will encounter throughout your maths degree, so the sooner you can embrace being stuck, the better.

Here are some suggestions for what to do when stuck.

- *Attempt a similar, easier exercise.* Often, by doing a simpler, related exercise you can get more of a feel for how the objects behave, and understand something else, which helps with the main exercise. At the very least, you will probably learn something else, or even what you know or don't know. If you can't solve the simpler exercise, then you probably won't be able to do the harder exercise, so you should go back to the notes or lectures and improve your knowledge and understanding.

- *Take a break.* Good problem-solving involves taking a break from the problem as well as consciously thinking about it. Our minds do a lot in the 'background', so taking a break from a problem is a productive use of your time. Rarely, do you sit down and solve a problem on the first go; usually, it takes time and multiple, spaced-out attempts (with your brain working in the background!), especially for more challenging problems and the further you go in university.

  By taking breaks you are trying to avoid working *passively*, which is what happens when you don't actively engage with the problem, or when you copy and listen, and how you start to work once you are tired out. You want to be working as *actively* as possible, which means doing the 'Getting started' techniques in an effortful and engaging manner, at least.

  A word of caution. Taking a break should not be seen as an easy way out; that would be working passively. Nor should you take a break without decently engaging in the problem (unless you don't currently have the energy to engage in the problem). If you haven't thought properly about a problem, then don't expect your brain to miraculously come up with a solution to it. If you find yourself getting nowhere after a few attempts and breaks, then you probably don't understanding things well enough and should get help.

- *Discuss.* You are on a course with hundreds of fellow maths students, all with different perspectives and approaches, and different strengths. Make the most of this. Everybody finds different things hard and different things easy. What you find difficult may be your friend's strength. You are expected, and indeed encouraged, to discuss problems and concepts with fellow students. University work is often too challenging to be done entirely by yourself, which reflects how things are in life - society wouldn't function if everybody worked as a sole individual. Plus, you will make friends and contacts through discussion, which is a great thing.

For ways to discuss mathematics with fellow students and instructors online, see 3.4 below.

- *Use the internet.* The internet is incredibly useful, in general, but I give this point reluctantly. For standard problems, you will easily find a solution online. Sure, finding it and essentially copying it may feel resourceful and an easy way forwards, but what benefit does that have on your learning? However, used for additional knowledge, perspectives, or hints and ideas, the internet can be very helpful.

  Here are some websites which may be useful for online help:

  1. This is a forum where many standard maths questions are posted and discussed: `https://math.stackexchange.com`.
  2. Khan Academy has many useful videos on topics covered in Year 1 and Year 2, such as linear algebra, calculus, and several variable calculus. (Unfortunately, however, it doesn't have any resources on proofs.) Link: `https://www.khanacademy.org/math`.
  3. Although it doesn't provide tutorials or notes on a specific question, Wikipedia is great for providing you with extra information and *intuition* for a certain topic. (Note, though, that Wikipedia maths articles can be quite dense and require at least Year 2-level maths to understand; but you can find material which suits your level.) Link: `https://www.wikipedia.org`.

- *Ask for a hint.* Lecturers are more than happy to offer a hint if a student asks a specific enough question. Summarise your reasoning as far as you have managed and explain what you are stuck on with moving forwards. Your lecturer will no doubt respond with a helpful hint.

  You can ask at the end of a lecture, workshop, or by email. Don't be afraid to send an email; lecturers are happy to receive such emails as it shows a student is engaging in the course and wanting to improve.

- *Name things.* It is surprising how helpful this can be. Assign each object a name, and that way it is easier to work with them and spotting patterns or relationships in their properties becomes easier. For example, labelling a set of vectors as $\{\vec{v}_1, ..., \vec{v}_n\}$ may make it easier to think about an argument involving span or linear independence.

3.3 Tips for improvement

You may feel - and this is totally normal - that you will never get better at solving problems or writing proofs, that it's simply not something you are good at. However, rest assured, you can and you will, so don't panic. Almost everybody finds proofs challenging at first. But this is also true when you try a new sport, yet you improve at it with practise. Writing proofs and solving problems in no different, except the muscle you are flexing is your brain. I felt as though I would never be good at proofs and that pure maths was not for me, yet I have done very well in my degree so far and will be doing postgraduate study in pure maths!

Here are some suggestions for getting better.

- *Try difficult problems.* Although you may not solve every difficult problem you try, your time doing this is very well spent. It is good to do questions of a variety of levels, but attempting suitably challenging ones will in many ways improve your understanding and knowledge the most. You will likely spend more time stuck on a challenging problem, and this time may seem unproductive, but if used well, it is not. By *well* I mean trying different approaches and learning why certain approaches may not work, familiarising yourself with definitions, thinking about the consequences of certain properties. If you do these things, instead of repeatedly trying the same thing, say, then you are actively improving what you know about the problem and all related theory (strengthening neural pathways), as well as refining what works and what doesn't and why or why not (refining neural pathways).

  Note, though, that some problems will be too far beyond your current ability to be worth your time. Time spent on such problems may actually be unproductive, as you will probably not get very far, it may knock your confidence, and the time would have been better spent on problems of a more appropriate level. Save those problems for once you have a greater understanding, as they will be the right level then.

- *Practise.* Of course, this goes without saying, but *how* should you practise? Certainly, all advice in this document points towards good practice. My other suggestions are: attempt to produce your best possible work for each proof, especially on hand-ins; be open to alternative ways of thinking about a problem, and trying lots of problems (but still quality over quantity).

Note - and this is *really* important - that no matter how you practise, you must at some stage receive feedback on your work - so by comparing your answer to a model solution, or by having a friend review your work, or receiving feedback from a workshop tutor. You may even want to discuss the way you practise, and get feedback on it.

- *Quality over quantity.* It is better to do less problems very thoroughly and - hopefully - well than more problems but to a lower standard. Sometimes however, you may not have time to spend all of your time on maybe one or two problems, and you may need to cover more problems in order to learn theory that is of immediate importance. A mix of drilling really deeply into a small number of problems (looking at the details) and covering a range of problems ('zooming out' from the details) is a good balance to be had.

3.4 Tips for writing proofs

Writing a proof is an art and not one that can be learnt in a formulaic manner or by rote; it takes experience, and lots of it. This is something you will be highly trained in at Edinburgh, so I won't go into great detail, but here are some suggestions on how to get better at writing proofs.

- *Take care and time.* A well-written proof takes, time and concentration to construct. It is advisable to leave yourself enough time to write your proof without being in a hurry; for complex proofs, this may run across several days. This time is well worth the effort because it involves organising your thoughts (which improves you understanding), improves your ability explain, to write concisely and precisely, and will help you to achieve good grades on hand-ins and assignments. Plus, you have done the hardest part by finding a proof, so it's your chance to show off your work!

- *Define all terms and notation.* You cannot argue in terms of undefined objects; you must first give their meaning. Unless notation is standard in the course, or in general, then it must be defined explicitly. Bottom line - when in doubt, define.[4]

- *Start by sketching an overall structure.* It is quite difficult, especially when you are still quite new to proofs, to write a proof in-full, first time. Even if you do, it is unlikely to be perfect; there will be wording or clarity than could be improved. I find it helpful to sketch out the overall structure of my argument, which means omitting the details and focusing only on the the main steps in the argument.

  For example, before writing the proof of Claim 2, I would sketch the structure of:

  - Prove by contradiction by suppose rational
  - rearrange and square to show that a must be even
  - using $a = 2k$ show that b must be even
  - Derive gcd contradiction

  Referring to a sketch like this makes writing the final proof easier. It is a bit like how construction lines makes it easier to draw a final artwork.

  Do not expect to write out a correct, elegant proof at the first sitting. Most proofs are the result of revision and rewriting.

3.5 Covid-19-related advice

Most of this document should be relevant regardless of being taught online or in-person, but certain things like discussion will be affected. Here is advice I have for online discussion, which will be of greater importance than normal, for obvious reasons with the virus restrictions. (Note though that this advice still stands under more normal circumstances.)

1. Create a group chat with the members of your workshop group and make use of any groups that are set up to support particular courses.
2. Use Piazza, which is an **anonymous** private forum containing only students and instructors from your course, and allows you to post questions, help other students out by answering their questions, and the instructors of the course can help by providing hints, corrections, or confirmation of correct answers. Personally, I find Piazza incredibly helpful, and I know other students do too.

[4]Except maybe for sets like $\{x : x \notin x\}$! See `https://en.wikipedia.org/wiki/Russell%27s_paradox`.

3. Arrange a video call with a fellow student or with a group. This is going to be more efficient and effective than talking about a question over text. Don't be afraid to ask this of somebody on your course; you are probably both in a similar position, so are happy to help each other out.

4. Contact people from your course via Facebook Messenger. You can message pretty much anyone from your course this way, and it is done by students all of the time. They don't need to be your best friend or even that familiar to message them.

4 Further Reading

- Lara Alcock's *How to Study for a Mathematics Degree* is a book packed with advice on not just proofs but on virtually all aspects of undergraduate mathematics. She also writes other very good books about mathematics.

- This is a brilliant article by Eugenia Cheng on many aspects of proofs: `https://deopurkar.github.io/teaching/algebra1/cheng.pdf`. She also writes very good books about mathematics.