FPM Formula Sheet

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Algebra

Defn 0.1.1: A function $f : X \to Y$ is called
Injective: $f(x) = f(y) \Rightarrow x = y$
Surjective: $\forall y \in im(f) : \exists x \in dom(f)$ such that $f(x) = y$.
Bijective: f is both *injective* and *surjective*

Defn 1.2.1: Group Axioms:
If $* : S \times S \to S$ is a binary operation on G, then G is a group if the following axioms are satisfied.
Closure: $g, h \in G \Rightarrow g * h \in G$.
Associativity: $g, h, k \in G \Rightarrow g * (h * k) = (g * h) * k$.
Identity: $\exists e \in G$ such that $\forall g \in G : g * e = e * g = g$.
Inverses: $\forall g \in G : \exists g^{-1} s.t. g^{-1} g = g * g^{-1} = e$.
Some examples of groups: $\mathbb{Z}, \mathbb{Q}$, or $\mathbb{R}$ under addition; S_n permutations of $\{1, \dots, n\}$; D_n symmetries of an n-gon, and $GL(n, G)$, the group of invertible matrices with entries in G ($|GL(n, G)| = \prod_{k=0}^{n-1}(|G|^n - |G|^k)$).

Thm 2.1.3: Subgroup Test:
A group H is a subgroup of G, written $H \leq G$, if:
S1: $H \neq \emptyset$
S2: $h, k \in H \Rightarrow hk \in H$.
S3: $h \in H \Rightarrow h^{-1} \in H$.

Thm 2.2.15: If G is cyclic and $H \leq G$ then H is cyclic.
Thm 2.2.16: $G \times H$ is cyclic $\Leftrightarrow hcf(|G|, |H|) = 1$

Thm 2.4: Lagrange's Theorem:
If $H \leq G$ the $|H|$ divides $|G|$.
Thm 2.3.8: $H \leq G \Rightarrow hH = H$. If $g_1, g_2 \in G, h \in H$ then the following three are equivalent statements:
- $g_1 H = g_2 H$;
- $\exists h \in H$ such that $g_2 = g_1 h$, and
- $g_2 \in g_1 H$.

Thm 2.4.2: $\forall g \in G : o(g)$ divides $|G|$, and $g^{|G|} = e$.
Thm 2.4.6: If $|G| = p$ for p prime then G is cyclic.
Col 2.4.7: If $|G| < 6$ then G is abelian.
Thm 4.3.2: **Cauchy's Theorem:**
Let G be a group, p be a prime. If p divides $|G|$, then G contains an element of order p.

Def 3.1.1: Homomorphisms:
A map $\varphi : G \to H$ is a homomorphism if $\varphi(xy) = \varphi(x)\varphi(y)$.
Lem 3.1.5: If φ is a homomorphism then $\varphi(e_G) = e_H$ and $\varphi(g^{-1}) = \varphi(g)^{-1}$.

Def 3.1.6: Let $\varphi : G \to H$ be a homomorphism.
$im(\varphi) := \{h \in H : h = \varphi(g)$ for some $g \in G\}$.
$ker(\varphi) := \{g \in G : \varphi(g) = e_h\} = \cap_{x \in X} Stab_G(x)$.
Def 3.1.7: Normal Subgroup:
N is normal to G, written $N \triangleleft G$, if $\forall g \in G : gN = Ng$.
Prop 3.1.8: If $\varphi : G \to H$ is a homomorphism, then $ker(\varphi) \triangleleft G$.
Prop 3.1.9: If $\varphi : G \to H$ is a homomorphism, then $ker(\varphi) = \{e\} \Leftrightarrow \varphi$ is injective $\Rightarrow G \cong im(\varphi)$.

Def 3.3.1: **Group Actions:**

An action of G on a set X is a map $\cdot : G \times X \to X$ such that:
$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ for $g_1, g_2 \in G$ and $x \in X$,
and where $\forall x \in X : e \cdot x = x$.
Def 4.1.1: $Stab_G(x) := \{g \in G : g \cdot x = x\}$.
Def 4.1.3: $Orb_G(x) := \{g \cdot x : g \in G\}$.
Def 4.1.7: An action is *transitive* if $\forall x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$. i.e. If G is a single orbit.
Ker: $Ker(\cdot) = \{g \in G : g \cdot x = x\}$. An action is *faithful* if $Ker(\cdot) = \{e\}$.

Thm 4.2.1: Orbit-Stabiliser Theorem
If G acts on X with $x \in X$ then $|G| = |Orb_G(x)| \times |Stab_G(x)|$.
Def 4.4.1: $Fix(g) := \{x \in X : g \cdot x = x\}$ it's worth noting that $Stab_G(x) \leq G$ but $Fix(g) \subseteq X$. **Thm 4.4.2:** the number of orbits on $X = \frac{1}{|G|} \sum_{g \in G} |Fix(g)|$.

Section 5: Conjugacy:
Let $(G, *)$ act on G with 'conjugacy' action $\cdot : G \times G \to G$, we define:
Action: $h \cdot g := hgh^{-1}$.
Centralizer: $C(g) := \{h \in G : gh = hg\} = Stab_G(g)$.
Centre: $C(G) := \{g \in G : \forall h \in G, gh = hg\} = \cap_{g \in G} C(g) = ker(\cdot)$.
Cor 5.1.6: $C(g) \leq G$, $\{e\}$ is always a conjugacy class and $C(G) \leq G$.
Thm 5.2.4: Two permutations in S_n are conjugate iff they have the same cycle type.
Thm 5.2.5: The number of elements of cycle type $1^{m_1}, 2^{m_2}, \ldots, n^{m_n}$ is $n! \div (m_1! \ldots m_n! 1^{m_1} 2^{m_2} \ldots n^{m_n})$.

Thm 5.3.3: Cayley's Theorem:
Every finite group is isomorphic to a subgroup of a symmetric group.

Analysis

Thm 1.2.3: Triangle Inequality: $|a + b| \leq |a| + |b|$, and $||a| - |b|| \leq |a - b|$

Def 1.3.2: Supremum:
A number $s = supE$ is a supremum of a set E if $\forall a \in E : a \leq s$ **and** $s \leq M$ for all upper bounds M of the set E.
Thm 1.3.5: Approximation Property:
If $E \subseteq \mathbb{R}$ has a supremum $supE$ then $\forall \varepsilon > 0$ we have that $supE - \varepsilon < a \leq supE$, for $a \in E$.

Def 2.1.1: Convergence:
A sequence (x_n) is said to converge to a if for every $\varepsilon > 0 : \exists N \in \mathbb{N}$ such that for all $n > N$:$|x - a| < \varepsilon$.
Thm 2.1.9: Every convergent sequence is bounded.

Thm 2.2.1: Squeeze Theorem:
If both (x_n) and (y_n) converge to a, and $\forall n : x_n \leq w_n \leq y_n$, then w_n converges to a also.
Thm 2.2.6: Divergence:
(x_n) is said to *diverge* to ∞ if for each $M \in \mathbb{R}$ there is $n \in \mathbb{N}$ such that for all $n > N$ we have $x_n > M$.

Def 2.3.1: Monotone:
A sequence (x_n) is monotone if it's increasing or decreasing, it's increasing if $\forall n : x_{n+1} \geq x_n$, and decreasing if $x_{n+1} \leq x_n$.
Thm 2.3.2: Monotone Convergence
If (x_n) is increasing (resp. decreasing) and bounded above (resp. below) then (x_n) is convergent. (tip: set $lim(x_n) = lim(x_{n+1})$ to find limit).

Def 3.2.5: limsup:
$\lim sup x_n = \lim_{N \to \infty} sup\{x_n : n > N\}$.

Def 2.3.8: Cauchy:
A sequence (x_n) is said to be *cauchy* if for all m, n we have that $|x_m - x_n| < \varepsilon$ for every $\varepsilon > 0$.
Thm 2.3.10: A sequence is convergent iff it's cauchy.

Thm 2.4.4: Every sequence of real numbers has a monotone subsequence.
Thm 2.4.5: Every bounded monotone sequence converges.
Thm 2.4.6: **Bolzano-Weierstrass:** Every bounded sequence of real numbers has a convergent subsequence.
Ross 11.3: If the sequence (s_n) converges, then every subsequence converges to the same limit.

Convergence Tests:

Divergence test: (a_n) diverges if $a_n \to a \neq 0$.
Telescopic: if (a_k) converges, $\sum_{k=1}^{\infty}(a_k - a_{k+1}) = a_1 - \lim_{k\to\infty} a_k$.
Geometric: $\sum_{k=0}^{\infty} x^k$ converges iff $|x| < 1$.
Comparison: If $a_k \leq b_k$ for all k then (a_k) converges if (b_k) does, and (b_k) diverges if (a_k) does.
Ratio Test: If $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to L$, then (a_n) converges if $L < 1$ and diverges if $L > 1$.

Thm 3.3.2: If $\sum a_n$ converges absolutely then it converges.

Def 4.1.1: **Continuity:**
A function f is continuous at x if for every sequence (x_n) that approaches x we have $\lim_{n\to\infty} f(x_n) = f(x)$.

Thm 4.1.6:
A function f is continuous at a if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Ross 17.5: If f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0.

Thm 4.2.2: **Extreme Value Thm:**
If $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ is continuous on I, then there exists points x_m, x_M such that $f(x_m) = inf\{f(x) : x \in I\}$ and $f(x_M) = sup\{f(x) : x \in I\}$.

Thm 4.2.4: **Intermediate Value Thm:**
If $f : I \to \mathbb{R}$ is continuous on I with $a, b \in I$ and $a < b$ then for every y_0 between $f(a)$ and $f(b)$ there exists x_0 such that $f(x_0) = y_0$.

Thm 2.4.9: If f is strictly increasing on I such that $im(f)$ is an interval then f is continuous.
Thm 2.4.10: If $f : [a, b] \to \mathbb{R}$ is strictly increasing and continuous then $f^{-1} : [f(a), f(b)] \to \mathbb{R}$ is too.

Def 5.1.1: **Differentiability:**
A function f is differentiable at a if $f'(a) = \lim_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists.

Thm 5.1.3: If f is differentiable at a then it's continuous.

Thm 5.3.1: **Rolle's Theorem:**
Suppose $a, b \in \mathbb{R}$ with $a < b$, and that f is continuous on $[a, b]$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists a point c where $f'(c) = 0$.

Thm 5.3.3: **Mean Value Theorem:**
If f is cts on $[a, b]$ and differentiable on (a, b) then there exists a point $c \in [a, b]$ such that $f(b) - f(a) = f'(c)(b - a)$.

Thm 5.4.4: **Inverse Function Theorem:**
If f is surjective and continuous on I and $a \in f(I)$, and if f' exists at the point $f^{-1}(a)$ (and is non-zero), then $(f^{-1})'(a) = \left[f'(f^{-1}(a))\right]^{-1}$.

Def 5.5.1 **Taylor's Polynomial:**
If $f : (a, b) \to \mathbb{R}$ is differentiable n times at $x_0 \in (a, b)$ then f can be approximated at the point x_0 by:
$f(x_0) \simeq P_n^{f,x_0}(x) = f(x_0) + \sum_{k=1}^{n} \dfrac{f^{(k)}(x_0)}{k!}(x - x_0)^k$.

Error term: The *error* of $P_n^{f,x_0}(x) \simeq f(x)$ in estimating $f(c)$ is given by $\dfrac{f^{(n+1)}(x_0)}{(n+1)!}(x - c)^{n+1}$.