Analysis Formula Sheet

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Convergence: Point-wise vs Uniform

Definition. A function f converges to L as $x \to a$ if $\forall \varepsilon > 0 : \exists \delta > 0 :$ such that $0 \neq |x-a| < \delta \implies |f(x) - L| < \varepsilon$.

Theorem (3.6)**.** *$f \to L$ as $x \to a$ on $I \setminus \{a\}$ iff for every sequence such that $x_n \to a$ we have $f(x_n) \to L$. **This is a good way of disproving convergence.***

Definition (Convergence)**.** A sequence of function $(f_n : E \to \mathbb{R})_{n\in\mathbb{N}}$ is said to converge

- **point-wise** if $\forall \varepsilon > 0 : \forall x \in E : \exists N : \forall n \geq N : |f - f_n| < \varepsilon$.
- **uniformly** if $\forall \varepsilon > 0 : \exists N : \forall n \geq N : \forall x \in E : |f - f_n| < \varepsilon$.

Theorem (7.9)**.** *If $f_n \to f$ uniformly and each f_n is continuous then f is continuous.*

~~*Proof.*~~ Let $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $n \geq N \Rightarrow |f_n - f| < \frac{\varepsilon}{3}$ (by *uniform convergence*). Let $\delta > 0$ so $|x - x_0| < \delta \Rightarrow |f_N(x_0) - f_N(x)| < \frac{\varepsilon}{3}$ (by *continuity of* f_N at x_0). So $|f(x) - f(x_0)| \leq |f - f_N| + |f_N(x_0) - f_N(x)| + |f_N(x_0) - f_N(x)| < \varepsilon$.

Theorem (7.10)**.** *If $f_n \to f$ uniformly then $\lim_{n\to\infty}\left(\int_a^b f_n(x)dx\right) = \int_a^b f(x)dx$.*

~~*Proof.*~~ Note that f is bounded on $[a,b]$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $n \geq N$ gives $|f(x) - f_n(x)| < \varepsilon$ for all $x \in [a,b]$. Choose step functions $\psi_P = \frac{\varepsilon}{|P|}$ for decreasing partitions P of $[a,b]$ so that $|f(x) - f_n(x)| < |\psi_n| < \frac{\varepsilon}{|P|}$. Then $0 \leq f < \psi_P$ for all P and $\int \psi_P - 0 \to 0$ so $f - f_n$ is integrable, giving that f is too, and $\lim(\int f_n) = \int \lim(f_n) = \int f$, as stated!

Theorem (Cauchy Criterion)**.** *The sequence $f_n \to f$ uniformly iff $\forall \varepsilon > 0 : \exists N \in \mathbb{N} : n, m > n \Rightarrow |f_n - f_m| < \varepsilon$.*

~~*Proof.*~~ Triangle inequality.

Theorem (Weierstrass M-Test:)**.** *If $(f_n : E \to \mathbb{R})_{n\in\mathbb{N}}$ and $\forall n : |f_n| \leq M_n$ with $\sum_{k=1}^{\infty} M_k < \infty \implies \sum_{k=1}^{\infty} f_k$ converges uniformly and absolutely.*

~~*Proof.*~~ Let $\varepsilon > 0$ and use Cauchy criterion to choose N s.t. $\forall m \geq n \geq N : \sum_{k=n}^{m} M_k < \varepsilon$. Then $|\sum_{k=n}^{m} f_k| \leq \sum_{k=n}^{m} |f_k| \leq \sum_{k=n}^{m} M_k < \varepsilon$.

Theorem (Dirichlet's Test)**.** *Suppose $f_k, g_k : E \to \mathbb{R}$ and $|\sum_{k=1}^{\infty} f_k| \leq M$ and $g_k \to 0$ uniformly then $\sum_{k=1}^{\infty} f_k g_k$ converges uniformly.*

~~*Proof.*~~ $|\sum_{k=m}^{n} f_k g_k| = \left|g_n \sum_{k=m}^{n} f_k + \sum_{k=m}^{n-1} (\sum_{i=k}^{m} f_k)(g_k - g_{k+1})\right| \leq 2Mg_n + 2M \sum_{k=m}^{n-1} (g_k - g_{k+1}) = 2Mg_m$.

Continuity: Point-wise vs Uniform

"Uniform continuity means $f_n(x) \to f(x)$ at roughly the same rate, for each x"

Definition. A function $f : E \to \mathbb{R}$ is uniformly continuous if $\forall \varepsilon > 0 : \exists \delta > 0 :$ such that $|x - a| < \delta$ and $a, x \in E \implies |f(x) - f(a)| < \varepsilon$.

Theorem (3.38)**.** *If $(x_n)_{n\in\mathbb{N}}$ is Cauchy and f is uniformly continuous then $(f(n))_{n\in\mathbb{N}}$ is Cauchy.*

~~*Proof.*~~ Cauchy $\Rightarrow \forall \delta > 0 : \exists N : \forall m, n > N$ we get $|x_m - x_n| < \delta$ then using uniform continuity $|f(x_m) - f(x_n)| < \varepsilon$.

Theorem (3.39)**.** *If f is continuous on a closed, bounded interval I then f is uniformly continuous on I.*

~~*Proof.*~~ Contradiction proof: Let $\varepsilon > 0$ and $\delta = 1/n$ and $|x_n - y_n| < 1/n$ but also $|f(x_n) - f(y_n)| \geq \varepsilon$. By Bolzano-Weierstrass both x_n and y_n have convergent subsequences $x_{n_k} \to x$ and $y_{n_j} \to y$ so that $|f(x) - f(y)| > \varepsilon$ ie $f(x) \neq f(y)$, but $|x_n - y_n| < 1/n \to 0$ so $x = y \Rightarrow f(x) = f(y)$, contradiction!

Theorem. *Suppose $f : (a,b) \to \mathbb{R}$ is continuous. Then f is uniformly continuous iff it can be extended continuously to $[a,b]$.*

Power Series

Definition (Radius of Convergence)**.** The radius of convergence (RoC), R, of a power-series $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ is such that $S(x)$ converges absolutely for $x - x_0 < R$ and diverges for $x - x_0 > R$. Thus

$$R = \sup\{r \geq 0 : (a_n r^n)_{n\in\mathbb{N}} \text{ is bounded}\}.$$

Theorem (Radius of convergence)**.** *Let $S(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$ be a power series with RoC R, then*

- *If $r = \left|\frac{1}{\limsup_{k\to\infty} a_k (x-x_0)^k}\right|^{1/k} < \infty$ then $R = r$ and $S(x)$ converges uniformly on $(x_0 - R, x_0 + R)$.*
- *If $r = \lim_{k\to\infty} \frac{|a_k|}{|a_{k+1}|}$ exists then $R = r$.*

~~*Proof.*~~ Root and ratio test, respectively.

Definition (Interval of Convergence)**.** The Interval of convergence of a power series $S(x)$ is the largest interval for which $S(x)$ converges.

Theorem (Abel's Theorem)**.** *If $S(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ converges on $[a,b]$ then $S(x)$ is continuous and converges uniformly on $[a,b]$.*

Theorem. *If $S(x) := \sum_{k=0}^{\infty} a_k (x - x_0)^k$ has RoC $R > 0$ then $S'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1}$ for $x \in (x_0 - R, x_0 + R)$.*

Theorem. *The two power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ and $\sum_{n=0}^{\infty} n a_n (x - x_0)^n$ have the same radius of convergence.*

Theorem. *If $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ has radius of convergence R then $S(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ is infinitely differentiable for $|x - x_0| < R$ with derivative $\sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$.*

Riemann Integration

Definition. The **characteristic function** of $E \subseteq \mathbb{R}$ is

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise} \end{cases}.$$

Definition. The **integral** of χ_E is

$$\int \chi_E = \text{length}(E).$$

Definition. We say that ϕ is a **step-function** if there are real numbers $x_0 < x_1 < \cdots < x_n$ such that

- $\phi(x) = 0$ for all $x < x_0$ and $x > x_n$ (this is called **bounded support**),
- $\phi(x)$ is constant on (x_{j-1}, x_j) $1 \leq j \leq n$.

Note that then $\phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1},x_j)}(x)$.

Definition. The **integral** of a step function $\phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1},x_j)}(x)$ is defined by

$$\int \phi = \int \phi(x) = \sum_{j=1}^n c_j \chi_{(x_{j-1},x_j)}(x)\,\mathrm{d}x = \phi(x) = \sum_{j=1}^n c_j(x_{j-1} - x_j).$$

Theorem. ***Integration is linear**: $\int(\alpha\phi + \beta\psi) = \alpha\int\phi + \beta\int\psi$.*

Definition. Let $f : \mathbb{R} \to \mathbb{R}$, we say f is **Riemann-integrable** if for all $\varepsilon > 0$ there exist step functions ϕ, ψ such that

$$\phi \leq f \leq \psi \quad \text{and} \quad \int \psi - \int \phi < \varepsilon.$$

Definition. Let f be Riemann integrable, then

$$\begin{aligned} \int f &= \sup\left\{ \int \phi \,:\, \phi \text{ is a step function and } \phi \leq f \right\} \\ &= \inf\left\{ \int \psi \,:\, \psi \text{ is a step function and } \psi \geq f \right\}. \end{aligned}$$

Theorem. *A function f is Riemann integrable iff there exists sequences of step functions $(\phi_n), (\psi_n)$ such that*

$$\phi_n \leq f \leq \psi_n \text{ and } \int \psi_n - \int \phi_n \to 0 \text{ as } n \to \infty.$$

Theorem. *Let f, g be Riemann integrable, then*

- *$af + bg$ is Riemann integrable with $\int(af+bg) = a\int f + b\int g$,*
- *if $f \geq 0$ then $\int f \geq 0$; if $f \leq g$ then $\int f \leq \int g$,*
- *$|f|$ is Riemann integrable and $|\int f| \leq \int |f|$,*
- *$\min\{f,g\}$ and $\max\{f,g\}$ are Riemann integrable, and*
- *fg is Riemann integrable.*

Theorem. *if $\tilde{f}(x) = f(x)$ for $x \in [a,b]$, where $\tilde{f} : [a,b] \to \mathbb{R}$ is continuous and $f(x) = 0$ if $x \notin [a,b]$ then f is Riemann integrable, and we define*

$$\int_a^b \tilde{f} = \int f.$$

Fundamental Theorem of Calculus

Theorem. *Let g be Riemann integrable and continuous for $x \in [a,b]$, then defining $G(x) = \int_a^x g$ we have that*

$$\frac{d}{dx} G(x) = g(x).$$

Theorem. *Suppose $f : [a,b] \to \mathbb{R}$ is continuously differentiable, then*

$$\int_a^b \frac{df}{dx} = f(b) - f(a).$$

Integrals and sequences/series

Theorem. *Suppose $(f_n : \mathbb{R} \to \mathbb{R})_{n\in\mathbb{N}}$ converges uniformly to $f : \mathbb{R} \to \mathbb{R}$ with each f_n Riemann-integrable, then f is Riemann integrable with*

$$\int f = \lim_{n\to\infty} \int f_n \text{ and thus,}$$
$$\int \sum_n f_n = \sum_n \int f_n.$$

Metric Spaces

"Metric spaces generalise our idea of *distance*."

Basic Ideas

Axiom (Metric Space)**.** A *metric space* $M = (X, \tau)$ is a set X and a function $\tau : X \times X \to X$ such that for all $x, y, z \in X$:

- **Positive-definite:** $\tau(x,y) \geq 0$ with $\tau(x,y) = 0$ iff $x = y$,
- **Symmetric:** $\tau(x,y) = \tau(y,x)$, and
- **Triangle Inequality:** $\tau(x,z) \leq \tau(x,y) + \tau(y,z)$.

Definition (Open and Closed Balls)**.** The open ball of radius r centred at x_0 of a metric space $M = (X,\tau)$ is $B_r(x_0) = \{x \in X : \tau(x,x_0) < r\}$. The closed ball is $\overline{B}_r(x_0) = \{x \in X : \tau(x,x_0) \leq r\}$. *These generalise open/closed intervals.*

Definition (Open and Closed sets)**.** A subset $V \subseteq X$ of a metric-space (τ, X) is said to be *open* if $\forall\varepsilon > 0 : \exists B_\varepsilon(x_0) \subseteq V$. A set $E \subseteq X$ is *closed* if $X \setminus E$ is open.

Theorem (10.14)**.** *Let $M = (X,\tau)$ be a metric space.*

- *A sequence in M can have at most one limit,*
- *if $x_n \to x$ and x_{n_k} is a subsequence of x_n then $x_{n_k} \to x$ also,*
- *every convergent sequence in M is bounded,*
- *every convergent sequence is Cauchy.*

Theorem (10.16)**.** *A set $E \subset X$ for metric space $M = (X,\tau)$ is closed iff every convergent sequence x_k in E satisfies $\lim_{k\to\infty}(x_k) \in E$.*

~~*Proof.*~~ Suppose (for contradiction) that $E \subseteq X$ is closed and $\lim_{k\to\infty} x_n \in X \setminus E$ which is open by definition of closed, then there is an N such that $\forall n \geq N$: $x_n \in X \setminus E$ by the definition of limits - contradiction! The other direction uses squeeze-theorem to show some sequence $\tau(x_k, x) \to 0$ with $x \in X \setminus E$ and $x_k \in E$, so that $x_k \to x$ giving a contradiction that x is and isn't in E.

Definition (Complete)**.** A metric space $M = (x,\tau)$ is *complete* if every Cauchy sequence $x_n \in X$ converges to some point in X.

Theorem (10.21)**.** *If (X,τ) is a complete metric space with $E \subseteq X$ then $(E, \tau|_E)$ is a complete metric space iff E is closed.*

~~*Proof.*~~ If E is complete and $x_n \to x \in E$ and so by Theorem 10.14(iv) the sequence $(x_k)_{k\in\mathbb{N}}$ is Cauchy, so by 10.16 E is closed. Suppose E is closed and $x_n \in E \subseteq X$ is Cauchy, then x_n is Cauchy (and hence convergent) in X since X is complete and since E is closed this limit must belong to E.

Limits and Continuity

Definition (Cluster Point)**.** A point $a \in X$ for metric space (X,τ) is a *cluster point of X* iff $\forall\delta > 0 : |B_\delta(a)| = \infty$. *This avoids the problem that $|x - y| < \delta$ may have no solutions with $x \neq y$.*

Definition (Convergence). Let a be a cluster point of metric space (X,τ) and $f : X\setminus\{a\} \to Y$ for some metric space (Y,ρ) then $f(x)$ converges to L if $\forall\varepsilon > 0 : \exists\delta > 0 : 0 < \tau(x,a) < \delta \implies \tau(f(x),L) < \varepsilon$.

Definition (Continuous). A function $f : E \subseteq X \to Y$ between metric spaces (X,τ) and (Y,ρ) is continuous at $x_0 \in E$ iff $\forall\varepsilon > 0 : \exists\delta > 0 : \tau(x,x_0) < \delta$ and $x \in E \implies \rho(f(x), f(x_0)) < \varepsilon$.

Theorem (10.28 and 10.29). *A function between metric spaces (X,τ) and (Y,ρ) is continuous iff for any sequence $x_n \to x \in X$ we have $f(x_n) \to f(x)$. Furthermore if $f : X \to Y$ and $g : Y \to Z$ are continuous then $\lim(g\circ f) = g(\lim(f))$.*

Interior, Closure and Boundary

Theorem (10.31). *Let $\{V_\alpha\}_{\alpha\in A}$ and $\{E_\beta\}_{\beta\in B}$ be a collection of open and closed sets, respectively:*

- *$\bigcup_{\alpha\in A} V_\alpha$ is open.*
- *If $|A|$ is finite then $\bigcap_{\alpha\in A} V_\alpha$ is open.*
- *$\bigcap_{\beta\in B} E_\beta$ is closed.*
- *If $|B|$ is finite then $\bigcup_{\beta\in B} E_\alpha$ is closed.*

~~*Proof.*~~ Just follow the definitions and use De-Morgan's laws.

Definition (Interior, Closure, Boundary). Let (X,τ) be a metric space and $E \subseteq X$

- The interior of E is $E^o := \bigcup\{V \subset E : V \text{ is open in } X\}$,
- the closure of E is $\overline{E} := \bigcap\{B \supseteq E : B \text{ is closed in } X\}$, and
- the boundary of E is $\partial E := \{x \in X : \forall r > 0\, B_r(x) \cap E \neq \emptyset \neq B_r(x) \cap (X\setminus E)\} = \overline{E}\setminus E^o$.

Thus $E^o \subseteq E$ is the largest open subset of E and $\overline{E} \supseteq E$ is the smallest closed superset of E. The fundamental theorem of calculus determines an integral based only on the endpoints of the domain, the boundary of a set generalises this.

Theorem (10.40). *Let $A,B \subseteq X$ for a metric space (X,τ), then*

$$(A\cup B)^o \supseteq A^o \cup B^o, \qquad (A\cap B)^o = A^o \cap B^o,$$
$$\overline{A\cup B} = \overline{A}\cup\overline{B}, \qquad \overline{A\cap B} \subseteq \overline{A}\cap\overline{B}.$$

Compactness

Compactness generalises the idea of a 'closed and bounded interval' which we use to prove the Extreme Value Theorem (EVT), thus the aim of this section is to prove EVT for more general metric spaces.

Definition (Covering). Let $\mathscr{V} = \{V_i\}_{i\in I}$ be a collection of subsets $V_i \subseteq X$ of a metric space (X,τ) with $E \subseteq X$.

- $\mathscr{V}$ **covers** E iff $E \subseteq \bigcup_{i\in I} V_i$,
- $\mathscr{V}$ is an **open cover** iff each V_i is open and $\mathscr{V}$ covers E, and
- $\mathscr{V}$ has a **finite subcover** iff $\exists I_0 \subseteq I : |I_0| < \infty$ such that $\{V_i\}_{I_0}$ covers $\mathscr{V}$.

Definition (Compact). A subset $H \subseteq X$ of a metric space is **compact** iff every open covering of H has a finite subcover, thus compact sets are those that can be covered by finitely many sets of arbitrarily small size.

Theorem (10.44-10.46). *Let $H \subseteq X$ for H closed and (X,τ) compact, then H is compact and thus is closed and bounded.*

Definition (Separable). A metric space (X,τ) is separable iff it contains a countable dense subset, ie $\forall a \in X : \exists (x_k)_{k\in\mathbb{N}} : x_k \to a$ as $k \to \infty$ with $x_k \in Z$ for some countable Z. This allows us to get a partial converse to Theorem 10.46.

Theorem (Lindelöf). *Let $E \subseteq X$ for separable metric space (X,τ). If $\mathscr{V}$ is an open cover of E then $\mathscr{V}$ has a finite subcover.*

Theorem (Heine-Borel). *Let (X,τ) be a separable metric space where every bounded sequence has a convergent subsequence and let $H \subseteq X$. Then H is compact $\Leftrightarrow$ H is closed and bounded.*

Connectedness

Definition (Connected). Let $U,V \subseteq X$ be open in a metric space (X,τ), they **separate** X iff $X = U\cup V$ and $U\cap V = \emptyset$. The space X is **connected** if no such separating U,V exist. (ie. the sets $(-\infty,\sqrt{2})\cup(\sqrt{2},\infty)$ separate $\mathbb{Q}$, so the rationals aren't connected).

Definition (Relatively open/closed). Let $E \subset X$ for metric space (X,τ), then $U \subseteq E$ is *relatively open* (resp. closed) if $\exists V \subseteq X : U = E\cap V$ and V open (resp. closed).

. . . .

Continuous Functions

Recall for this section that function $f : (X,\tau) \to (Y,\rho)$ is *continuous* at $a \in X$ if $\forall\varepsilon > 0 : \exists\delta > 0 : \tau(x,a) < \delta \Rightarrow \rho(f(x), f(a)) < \varepsilon$. In other words $B_\delta(a) \subseteq f^{-1}\big(B_\varepsilon(f(a))\big) \subseteq X$. This leads to

Theorem (Continuity). *A function $f : X \to Y$ is continuous iff $\forall V \subseteq Y$ open: $f^{-1}(V)$ is open.*

Theorem (10.61 and 10.62). *If $H \subseteq X$ is compact and $f : H \to Y$ is continuous then $f(X)$ is compact. If E is connected in X and $f : E \to Y$ is continuous then $f(E)$ is connected.*

Theorem (Extreme Value Theorem). *Let $\emptyset \neq H \subset X$ be a compact subset of a metric space (X,τ) with $f : H \to \mathbb{R}$ continuous, then $M := \sup\{f(x) : x \in H\}$ and $m := \inf\{f(x) : x \in H\}$ are finite real numbers and $\exists x_M, x_m \in H : f(x_m) = m$ and $f(x_M) = M$.*

~~*Proof.*~~ Since H is compact so is $f(H)$ by 10.61, so by 10.64 $f(H)$ is closed and bounded, thus M is bounded (ie finite). By the approximation property let $x_k \in H$ be so that $f(x_k) \to M$ so that $x_M = \lim_{k\to\infty}(x_k)$ since $f(H)$ is closed and thus contains its limit points.

Theorem (10.64). *If H is compact and $f : H \to Y$ is injective and continuous then f^{-1} is continuous on $f(H)$.*

Contraction Mappings

Definition. Let (X,d) be a metric space. A function $f:(X,d)\to(X,d)$ is a **contraction mapping** if there exists an $\alpha\in(0,1)$ such that

$$d(f(x),f(y))\le\alpha d(x,y)\quad\text{for all } x,y\in X.$$

Brower's Fixed point theorem

Theorem. *If (X,d) is a complete metric space with contraction $f:(X,d)\to(X,d)$ then there exists a unique 'fixed point' x such that*

$$f(x)=x$$

Proof. Pick $x_0\in X$ and let $x_{n+1}=f(x)$, then $d(x_{n+1},x_n)=d(f(x_n),f(x_{n-1})\le\alpha d(x_n,x_{n-1})\le\alpha^n d(x_1,x_0)$. Thus if $m>n$ then

$$\begin{aligned}d(x_m,x_n)&\le d(x_m,x_{m-1})\dots d(x_{n+1},x_n)\\&\le(\alpha^{m-1}\dots\alpha^n)d(x_1,x_0)\\&\le\frac{\alpha^n}{1-\alpha}d(x_1,x_0)\to 0\text{ as }n\to\infty.\end{aligned}$$

Hence $(x_n)_{n\in\mathbb{N}}$ is Cauchy thus (by completeness) is convergent to some x. Now contraction maps are continuous, so

$$f(x)=\lim_{n\to\infty}x_n=\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}x_{n+1}=x,$$

giving our fixed point, which is unique by the uniqueness of limits. □

Differential Equations

Our aim is to find conditions under which the system

$$\frac{dx}{dt}=F(x,t),\quad x(0)=A$$

has a *unique* solution $x(t)$ for $|t|<s$.

Definition. A function $F(x,t)$ satisfies the **Lipshitz condition in** x if for $A\in\mathbb{R}$ where $F:[A-\rho,A+\rho]\times[-r,r]\to\mathbb{R}$ is continuous for $\rho,r>0$, and

$$\forall x,y\in[A-\rho,A+\rho]:\forall t\in[-r,r]:\exists M>0:|F(x,t)-F(y,t)|\le M|x-y|.$$

Theorem (Picard)**.** *Suppose $F(x,t)$ satisfies the Lipschitz condition in x, then*

$$\exists s>0:\frac{dx}{dt}=F(x,t),\quad x(0)=A \tag{1}$$

has a unique solution $x(t)$ for $|t|<s$.

Proof. Consider the mapping T given by

$$T(x)(t)=A+\int_0^t F\big(x(u),u\big)\mathrm{d}u,$$

this is a contraction mapping of the (complete) space of continuous functions $\mathscr{C}([A-\rho,A+\rho])$ with metric $d_\infty(f,g)=\sup_{|u|\le s}|f(u)-g(u)|$ for a *sufficiently small* s. Choosing such an s gives, by Banach's fixed-point theorem, that there is a 'fixed point', i.e. an element $f\in\mathscr{C}([A-\rho,A+\rho])$ where $T(f)(t)=f(t)$, which is then a solution to (1) by the fundamental theorem of calculus. □

Useful Formula

Various Equations

Theorem (De-Morgan's Law)**.** *If for any $\alpha \in A$ for indexing set A we have $E_\alpha \subseteq X$ then*

$$X \setminus \bigcup_{\alpha \in A} E_\alpha = \bigcap_{\alpha \in A} E_\alpha \quad \text{and} \quad X \setminus \bigcap_{\alpha \in A} E_\alpha = \bigcup_{\alpha \in A} E_\alpha.$$

Axiom (Completeness)**.** If $A \subset \mathbb{R}$ then $\sup(A) \in \mathbb{R}$ exists.

Theorem (Nested Interval)**.** *If $(I_n)_{n\in\mathbb{N}}$ is a sequence of non-empty bounded intervals then $\bigcap_{n\in\mathbb{N}} I_n \neq \emptyset$, specifically if $|I_n| \to 0$ then $I_n \to \{x\}$, a singleton.*

Theorem. *The metric space $(\mathbb{R}, |\cdot|)$, ie $\mathbb{R}$ under the Euclidean metric, is complete ("Cauchy $\implies$ convergent").*

~~*Proof.*~~ Suppose that $(x_n)_{n\in\mathbb{N}}$ is Cauchy so that $\forall \varepsilon > 0 : \forall m, n > N : |x_m - x_n| < \varepsilon$. Choose $\varepsilon = 1$ and N so that $|x_N - x_m| < 1$ for all $m > N$ giving by the triangle inequality that $|x_m| < 1 + |x_N|$. Thus $(x_n)_{n\in\mathbb{N}}$ is bounded by $M := \max\{|x_1|, \dots, |x_{N-1}|, |x_N| + 1\}$, then use Bolzano-Weierstrass.

Theorem (Bernoulli's inequality)**.** *Let $x \in [-1, \infty)$. Then $0 < \alpha \leq 1 \implies (1+x)^\alpha \leq 1 + \alpha x$ and $\alpha \geq 1 \implies (1+x)^\alpha \geq 1 + \alpha x$.*

~~*Proof.*~~ Let $f(t) = t^\alpha$ then $f'(t) = \alpha t^{\alpha-1}$ then by Mean Value Theorem $\exists c \in [1, 1+x] : f(1+x) - f(1) = \alpha x c^{\alpha-1}$. Then split into cases $x > 0$ and $-1 \leq x \leq 0$.

FPM Revision

Definition. A function f is continuous at a iff $\forall \varepsilon > 0 : \exists \delta > 0$ such that $|x - a| < \delta \implies |f(x) - f(a)| < \varepsilon$.

Theorem (Extreme Value)**.** *If I is a closed, bounded interval then any $f : I \to \mathbb{R}$ is bounded and f will attain these upper and lower bounds.*

~~*Proof.*~~ Assume this is false so that $|f(x_n)| > n$ for some $n \in \mathbb{N}$, by Bolzano-Weierstrass there is a subsequence $x_{n_k} \to a$, so that $|f(x_{n_k})| > n_k$ will have the same limiting behaviour as $|f(x_n)|$, but then $f(a) \to \infty$ - a contradiction of Bolzano-Weierstrass! So f is bounded.

Theorem (Intermediate Value)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is continuous, if y_0 lies between $f(a)$ and $f(b)$ then $\exists x_0 \in (a,b) : f(x_0) = y_0$.*

~~*Proof.*~~ WLOG assume $f(a) < y_0 < f(b)$. Consider $E := \{x \in [a,b] : f(x) < y_0\}$, by the completeness axiom $x_0 := \sup E$ exists, so choose a sequence so that $x_n \to x_0$ then by continuity $f(x_0) = \lim f(x_n) \leq y_0$. For contradiction, assume that $f(x_0) < y_0$ then $0 < y_0 - f(x_0)$ is continuous on $[a,b)$, so $\exists x_1 > x_0$ such that $y_0 - f(x_1) > \varepsilon > 0$ so that $\sup E < x_1 \in E$, a contradiction!

Theorem (Rolle)**.** *Suppose that $a < b$. If f is continuous on $[a,b]$ and differentiable on (a,b) with $f(a) = f(b)$ then $\exists c \in (a,b) : f'(c) = 0$.*

~~*Proof.*~~ Look at the maximum $\forall x \in [a,b] : f(x) \leq f(c) := M$, and let $\varepsilon > 0$ so that $f(c - \varepsilon) - f(c) \leq 0$ and $f(c+\varepsilon) - f(c) \geq 0$, then use intermediate value theorem to get $f'(c) \lim_{\varepsilon \to 0} \left(\frac{f(c-\varepsilon) - f(c)}{\varepsilon}\right) = 0$.

Theorem (Mean Value)**.** *If f, g are continuous on $[a,b]$ and differentiable on (a,b) then $\exists c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$ and $g'(c)\big(f(b) - f(a)\big) = f'(c)\big(g(b) - g(a)\big)$.*

Theorem (2.8)**.** *Every convergent sequence is bounded.*

~~*Proof.*~~ Let $M := \max\{|x_1|, |x_2|, \dots, |x_N|\}$, then use $n > N \Rightarrow |x_n - a| < 1 \Rightarrow |x_n| < 1 + |a|$.

Theorem (Squeeze Theorem)**.** *Suppose $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ are sequences in $\mathbb{R}$, then if $x_n, y_n \to a$ and $\forall n : x_n \leq w_n \leq y_n$ then $w_n \to a$ also.*

~~*Proof.*~~ Just use $a - \varepsilon < x_n \leq w_n \leq y_n < a + \varepsilon$.

Theorem (Comparison Test)**.** *Suppose $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent with $\forall n : x_n \leq y_n$ then $\lim_{n\to\infty}(x_n) \leq \lim_{n\to\infty}(y_n)$.*

~~*Proof.*~~ Proof by contradiction: assume that $\forall n : x_n \leq y_n$ and $\lim_{n\to\infty}(x_n) > \lim_{n\to\infty}(y_n)$. Let $\varepsilon := \frac{x-y}{2} > 0$ and choose N so that $\forall n \geq N : |x - x_n|, |y - y_n| < \varepsilon$. Then $x_n > x - \varepsilon = y + \varepsilon > y_n$ so $\exists n : x_n > y_n$, contradiction!

Theorem (Monotone Convergence)**.** *If $(x_n)_{n\in\mathbb{N}}$ is increasing and bounded above then it converges.*

~~*Proof.*~~ Use approximation property for the supremum $a := \sup\{x_n : n \in \mathbb{N}\}$ (which exists by the completeness axiom) to get $\forall \varepsilon > 0, n > N :\ a - \varepsilon < x_N \leq a$ then use the squeeze theorem.

Theorem (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

~~*Proof.*~~ Choose $a, b \in \mathbb{R}$ such that $\forall n : x_n \in [a,b] =: I_0$, then divide I_0 into two sets $I_0 = [a, \frac{b-a}{2}]$. Define I_1 as the one of these halves that contain infinitely many x_n, and repeat the split so that $|I_n| \to 0$, then by the nested interval theorem $\lim_{n\to\infty} I_n = \{x\}$, the limit of a subsequence.

Theorem (2.36)**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. Then $x_n \to x$ as $n \to \infty$ if and only if $\limsup x_n = \liminf x_n = x$.*

~~*Proof.*~~ Use that $\inf(x_n) \leq x_n \leq \sup(x_n)$ and squeeze-theorem.

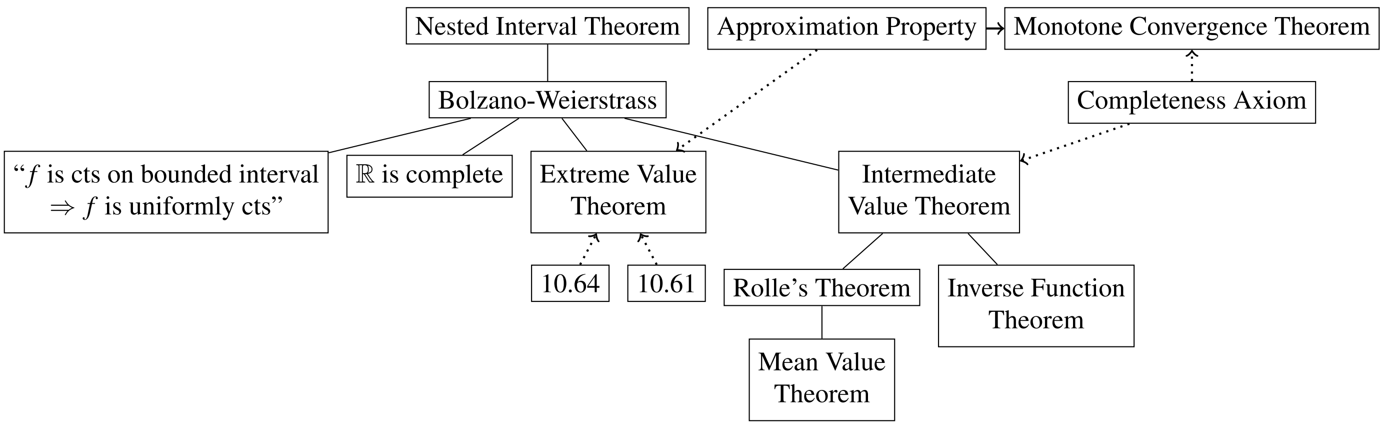
Definition. The partial sum of order n of a sequence $(a_n)_{n\in\mathbb{N}}$ is given by $s_n := \sum_{i=1}^n a_i$. The infinite sum $\sum_{i=1}^\infty a_i$ converges if $(s_n)_{n\in\mathbb{N}}$ does.

Definition. A series $\sum_{i=0}^\infty a_n$ converges *absolutely* if $\sum_{i=0}^\infty |a_n|$ converges.

Theorem (Convergence Tests)**.** *Let $(a_n)_{n\in\mathbb{N}}$ be a sequence and $s_n := \sum_{i=0}^n a_i$ and $s_\infty := \sum_{i=0}^\infty a_i$, and also let $r_\infty := \sum_{i=0}^\infty b_i$, then*

- ***Divergence test:*** *If s_∞ converges then $a_n \to 0$ as $n \to \infty$.*
- ***Telescoping:*** *$\sum_{i=1}^\infty (a_i - a_{i+1}) = a_1 - \lim_{n\to\infty} a_n$.*
- ***Geometric Series:*** *$|x| < 1 \iff \sum_{i=N}^\infty x^n = \frac{x^N}{1-x}$.*
- ***Cauchy Criterion:*** *s_∞ converges iff $\forall \varepsilon > 0 : \exists N : m \geq n \geq N \Rightarrow |s_n| < \varepsilon$.*
- ***Integral Test:*** *If $f > 0$ is decreasing then $\sum_{i=0}^\infty f(k)$ converges iff $\int_1^\infty f(x)\,dx < \infty$.*
- ***p-test:*** *$\sum_{i=1}^\infty \frac{1}{i^p}$ converges iff $p > 1$.*
- ***Comparison Test:*** *If $\forall i : 0 \leq a_i \leq b_i$ then r_∞ converges $\Rightarrow s_\infty$ converges, and s_∞ diverges $\Rightarrow r_\infty$ diverges.*
- ***Limit Comparison Test:*** *Let $L := \lim_{n\to\infty} \frac{a_n}{b_n}$ then $L > 0 \Rightarrow r_\infty$ converges iff s_∞ does. If $L = 0$ then r_∞ converges $\Rightarrow s_n$ converges.*
- ***Root Test:*** *Let $d := \limsup_{k\to\infty} |a_k|^{1/k}$. Then $d < 1 \Rightarrow s_k$ converges absolutely, and if $d > 1$ then it diverges.*
- ***Ratio Test:*** *Let $d := \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$ then $d < 1 \Rightarrow s_k$ converges absolutely, and if $d > 1$ then it diverges.*

Dependency tree of theorems



Probably non-examinable

Theorem (4.32)**.** *Suppose I is an interval and $f : I \to \mathbb{R}$ is injective and continuous on I, then $J := f(I)$ is an interval on which f^{-1} is monotone, and f is monotone on I.*

~~*Proof*~~. Suppose $a, b \in I$ and $c, d \in J$ with $f(a) = c < f(b) = d$, and let $y_0 \in (c, d)$, then by Intermediate value theorem (since f is continuous) $\exists x_0 \in (a, b) : y_0 = f(x_0)$. To summarise, $y_0 \in (c, d) \implies y_0 \in J$ so J is an interval. To prove f is monotone use contradiction; assume f isn't monotone then use intermediate value theorem again to derive that $f(c) < f(a) < f(b)$ or $f(a) < f(b) < f(c) \implies \exists x_1 \in (a, b) : f(x_1) = f(a)$ or $f(x_1) = f(b)$, a contradiction!

Theorem (Inverse Function Theorem)**.** *If I is an interval and $f : I \to \mathbb{R}$ is injective and continuous with $b = f(a)$ and $f'(a)$ exists, then $\frac{d}{dt} f^{-1}(b) = \frac{1}{f'(a)}$. In other words $\frac{dy}{dx} = \frac{1}{dx/dy}$.*

~~*Proof*~~. Theorem 4.32 gives that f is monotone, so we can fix the intervals of f and f^{-1}. Then use that f is continuous to swap f^{-1} with differentiation limits:

$$\lim_{h \to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} = \lim_{x \to a} \frac{x - a}{f(x) - f(a)} = \frac{1}{f'(x)},$$

where $x := f^{-1}(b+h)$ and $b = f^{-1}(a)$ by assumption so that $f(x) - f(a) = (b+h) - b = h$.

Definition. A series $\sum_{k=0}^{\infty} b_k$ is a *rearrangement* of $\sum_{k=0}^{\infty} a_k$ iff there is an injective function so that $b_{f(k)} = a_k$.

Theorem (6.27)**.** *If $\sum_{k=0}^{\infty} a_k$ converges absolutely and $\sum_{k=0}^{\infty} b_k$ is a rearrangement then $\sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} a_k$.*

Theorem (Riemann)**.** *If $x \in \mathbb{R}$ and $\sum_{k=0}^{\infty} a_k$ is conditionally (but not absolutely) convergent then there is a rearrangement of $\sum_{k=0}^{\infty} a_k$ so that $\sum_{k=0}^{\infty} b_k = x$.*