Complex Variables Formula Sheet

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Chapter One - Holomorphicity

Theorem (Triangle Inequality).

$$|z+w| \leq |z|+|w|$$
$$||z|-|w|| \leq |z-w|$$

Definition. The **argument** of $z \in \mathbb{C}$ is $\arg(z) := \{\theta : z = |z|e^{i\theta}\}$, the **principle argument** is $\operatorname{Arg}(z) \in \arg(z) \cap (-\pi, \pi]$ and is *unique*.

Theorem (1.1.19). *Let $z, w \in \mathbb{C}$ be non-zero. Then $\arg(zw) = \arg(z) + \arg(w)$ and $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)$.*

Definition. The **open (resp. closed) ε-disk** centred at z_0 is $D_\varepsilon(z_0) := \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$ (resp. $\overline{D_\varepsilon(z_0)} := \{z \in \mathbb{C} : |z - z_0| \leq \varepsilon\}$). The **puntured disk** is $D'_\varepsilon(z_0) := D_\varepsilon(z_0) \setminus \{z_0\}$.

Definition. A subset $D \subseteq \mathbb{C}$ is **open** if $\forall z \in D : \exists \varepsilon > 0 : D_\varepsilon(z) \subseteq D$ (or it's a union of open disks) and is **closed** if $\mathbb{C} \setminus D$ is open. If $z \in D$ is open we say D is a **neighbourhood** of z.

Definition. Let $S \subseteq \mathbb{C}$ then $z_0 \in \mathbb{C}$ is a **limit-point** of S if $\forall \varepsilon > 0 :$ $D'_\varepsilon(z_0) \cap S \neq \emptyset$. If L_S is the set of limit points of S then $\overline{S} := S \cup L_S$ is the **closure** of S.

Theorem (1.2.9). *A complex sequence z_n **converges** iff $\operatorname{Re}(z_n)$ and $\operatorname{Im}(z_n)$ converge.*

Theorem. *The complex plane $\mathbb{C}$ is **complete**, namely z_n is convergent $\Leftrightarrow z_n$ is **Cauchy**.*

Theorem (Bolzano-Weierstrass). *If z_n is a bounded sequence then it has a convergent subsequence.*

Theorem (1.3.3). *Let $f : S \subseteq \mathbb{C} \to \mathbb{C}$ and $z_0 = x_0 + iy_0 \in \overline{S}$ and $z = x + iy, a_0 \in \mathbb{C}$, then $\exists u(x,y), v(x,y)$ such that $f(z) = u(x,y) + iv(x,y)$. Then $a_0 = \lim_{z \to z_0} f(z)$ iff $\operatorname{Re}(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y)$ and $\operatorname{Im}(z) = \lim_{(x,y)\to(x_0,y_0)} v(x,y)$.*

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous** if $f^{-1}(U)$ is open for all $U \subseteq \mathbb{C}$, equally if $\forall \varepsilon > 0 : \exists \delta > 0 : |f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$ then f is continuous at z_0.

Theorem. *If $S \subseteq \mathbb{C}$ is **compact** (i.e. closed and bounded) then $f(S)$ is compact.*

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is **differentiable** at z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = 0$. Furthermore **differentiability gives continuity**.

Theorem. *Let $z_0 = x_0 + iy_0 \in \mathbb{C}$ and $U \subseteq \mathbb{C}$ be a neighbourhood of z_0 with $f : U \to \mathbb{C}, f = u + iv$ differentiable at z_0. The **Cauchy-Riemann equations** are*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is **holomorphic** at z_0 if it's differentiable on an open neighbourhood of z_0.

Definition. If $u : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ is harmonic (i.e. $u_{xx} + u_{yy} = 0$) and $f = u + iv$ is holomorphic, then v is the **harmonic conjugate** of u.

Definition. The **complex exponential** is $\exp(z) = e^x(\cos(y) + i\sin(y))$ where $z = x + iy$. It is holomorphic on all of $\mathbb{C}$ (prop 1.6.2).

Theorem. *Let $z, w \in \mathbb{C}$, then*

$$\exp(z+w) = \exp(z)\exp(w) \quad \text{and} \quad \exp(z + 2\pi i) = \exp(z).$$

Definition. The **complex logarithm** for $z \in \mathbb{C}$ is $\log(z) := \{w \in \mathbb{C} : \exp(w) = z\}$.

Theorem (1.7.3). *Let $z, w \in \mathbb{C}$, then*

$$\log(z) = \ln|z| + i\arg(z), \quad \log(zw) = \log(z) + \log(w),$$
$$\text{and } \log(1/z) = -\log(z).$$

Definition. The **principle logarithm** is $\operatorname{Log}(z) := \ln|z| + i\operatorname{Arg}(z)$.

Definition. A **branch cut** is $L_{z_0,\theta} := \{z \in \mathbb{C} : z = z_0 + re^{i\theta}, r \geq 0\}$, giving the **cut plane** $D_{0,\pi} := \mathbb{C} \setminus L_{0,\pi}$. If we let $\operatorname{Arg}_\theta(z) := \arg(z) \cap (\theta, \theta + 2\pi]$ then $\operatorname{Log}_\theta := \ln|z| + i\operatorname{Arg}_\theta(z)$.

Theorem (1.7.10). *Let $\theta \in \mathbb{R}$ and $U \subseteq \mathbb{C}$ with $g : U \to \mathbb{C}$ holomorphic, then $Log(g(z))$ is holomorhpic on $U \cap g^{-1}(D_{0,\theta})$. Particularly, if g is hol. on $\mathbb{C}$ then $Log(g(z))$ is holomorhpic on $g^{-1}(D_{0,\theta})$.*

Chapter Two - Möbius transformations

Theorem. *Let $f : U \to \mathbb{C}$ be holomorphic with $U \subseteq \mathbb{C}$ open, then f is **conformal** (preserves angles) at $z_0 \in U$ iff $f'(z_0) \neq 0$.*

Definition. A **Möbius transformation** is any function of the form $f(z) = \frac{ax+b}{cz+d}$ where $ad - bd \neq 0$. This forms a group $\mathscr{M} \cong \mathrm{SL}(2;\mathbb{C})$.

Definition. The **Riemann sphere** is $S^2 := \{(X,Y,Z) \in \mathbb{R}^3 : X^2 + Y^2 + Z^2 = 1\}$. The **extended complex plane** is $\tilde{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ where $\frac{1}{\infty} = 0$ and $z \cdot \infty = \infty \cdot z = \infty = \frac{1}{0}$ for any $z \in \mathbb{C}$.

Theorem. *Möbius transformations are holomorphic and conformal on $\tilde{\mathbb{C}}$.*

Theorem. *Let $z = x + iy$. **Stereographic projection** is the pair of inverse bijections ($\varphi : S^2 \to \tilde{\mathbb{C}}, \psi : \tilde{\mathbb{C}} \to S^2$) given by:*

$$\varphi(X,Y,Z) := \frac{X + iY}{1 - Z} \quad \text{and} \quad \psi(z) := \left(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)$$

Theorem (2.4.3). *A Möbius transformation maps **circlines** (circles and lines) to circlines.*

Theorem. *The **cross-ratio** is the unique Möbius transformation which sends $(z_2, z_3, z_4) \mapsto (1, 0, \infty)$. The image of z under this is given by*

$$[z, z_2, z_3, z_4] = \frac{z - z_3}{z - z_4}\frac{z_2 - z_4}{z_2 - z_3}.$$

Theorem (2.5.7). *Let M be a Möbius transformation, then $[Mz_1, Mz_2, Mz_3, Mz_4] = [z_1, z_2, z_3, z_4]$.*

Chapter Three - Complex integration

Definition. Let $[a,b] \subseteq \mathbb{R}$ be a closed interval and $f : [a,b] \to \mathbb{C}$ be of the form $f = u + iv$, then f is **integrable** if u and v are in the real sense. Then $\int_{[a,b]} f(t)dt = \int_{[a,b]} u(t)dt + i\int_{[a,b]} v(t)dt$.

Theorem (3.1.2)**.** *Integration in $\mathbb{C}$ is linear, and $\int_a^b \frac{dF}{dt}dt = F(b) - F(a)$. One estimate for integration is $\left|\int_a^b f(t)dt\right| \leq \int_a^b |f(t)|dt$.*

Definition. A **parametrized curve** Γ from z_0 to z_1 (distinct) is a continuous function $\gamma : [t_0,t_1] \to \mathbb{C}$ with $\gamma(t_0) = z_0$ and $\gamma(t_1) = z_1$. It is **regular** if $\gamma'(t)$ exists, is continuous and non-zero.

Definition. Let Γ be a regular curve and $f : \Gamma \to \mathbb{C}$ continuous. The **integral of** f **along** Γ is $\int_\Gamma f(z)dz = \int_{t_0}^{t_1} f(\gamma(t))\gamma'(t)dt$.

Definition. The **arc-length of** Γ is $l(\Gamma) := \int_{t_0}^{t_1} |\gamma'(t)|dt$.

Theorem (3.2.9,**ML Lemma**)**.** *Let Γ be regular and $f : \Gamma \to \mathbb{C}$ continuous, then*

$$\left|\int_\Gamma f(z)dz\right| \leq \max_{z\in\Gamma} |f(z)|l(\Gamma)$$

Definition. $D \subseteq \mathbb{C}$ **is a domain** if it's open and $\forall z,w \in D : \exists\Gamma$, a contour connecting z to w.

Theorem. ***Fundamental Theorem of Calculus:*** *Let D be a domain and $\Gamma \subseteq D$ a contour connecting $z_0, z_1 \in D$ and $F' = f$ then $\int_\Gamma f(z)dz = F(z_1) - F(z_0)$.*

Definition. Let $\Gamma \subseteq D$ be a contour in domain $D \subseteq \mathbb{C}$, Γ is a **closed contour** if it has equal endpoints ($\gamma(t_0) = \gamma(t_1)$).

Theorem. ***Path-independence:*** *Let $D \subseteq \mathbb{C}$ be a domain with continuous $f : D \to \mathbb{C}$ then the following are equivalent:*

- *f has an anti-derivative F on D,*
- *$\int_\Gamma f(z)dz = 0$ for all closed contours $\Gamma \subseteq D$,*
- *all contour integrals $\int_\Gamma f(z)dz$ are independent of path, thus depend only on the end-points.*

Definition. A contour Γ is **simple** if it has no self-intersections, if it is also closed then we call it a **loop**. A loop is **positively-oriented** if a parametrisation γ goes around anti-clockwise.

Definition. Let Γ be a loop, then $\text{Int}(\Gamma)$ is the interior, and $\text{Ext}(\Gamma)$ is the exterior so that $\mathbb{C} = \text{Int}(\Gamma) \cup \Gamma \cup \text{Ext}(\Gamma)$.

Definition. A domain D is **simply-connected** if for any loop Γ : $\text{Int}(\Gamma) \subseteq D$.

Theorem. ***Cauchy-Integral:*** *Let Γ be a loop and f be holomorphic inside and on Γ, then the following hold:*

$$\int_\Gamma f(z)dz = 0,$$
$$\frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z - z_0}dz = f(z_0),$$
$$\frac{n!}{2\pi i}\int_\Gamma \frac{f(w)}{(w-z)^{n+1}}dz = f^{(n)}(z).$$

Theorem (3.4.11)**.** *Let Γ be a loop not passing through z_0, then*

$$\int_\Gamma \frac{1}{z - z_0}dz = \begin{cases} 2\pi i & \text{if } z_0 \in \text{Int}(\Gamma) \\ 0 & \text{otherwise} \end{cases}.$$

Theorem (3.4.12)**.** *Let Γ_1, Γ_2 be loops with f holomorphic on both then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$, i.e. the two loops can freely be **deformed** into each other.*

Theorem (3.5.2)**.** *Let f be holomorphic on a domain D, then f has infinitely many derivatives, all of which are holomorphic.*

Theorem (Morera)**.** *Let $D \subseteq \mathbb{C}$ be a domain and f is continuous with $\int_\Gamma f(z)dz = 0$ for all loops Γ, then f is holomorphic on D.*

Theorem. *Let $f : \overline{D_R}(z_0) \to \mathbb{C}$ be holomorphic and bounded by M. Then*

$$|f^{(n)}(z_0)| \leq \frac{n!M}{R^n}.$$

Theorem. ***Liouville:*** *Let f be holomorphic on $\mathbb{C}$ and bounded, then f is constant.*

Theorem. ***Maximum modulus principle:*** *Let $D \subseteq \mathbb{C}$ be a domain on which f is holomorphic and bounded by M. If f achieves its maximum inside D the f is constant on D.*

Chapter Four - Series expansions

Theorem. *Convergence tests*

- ***Comparison test:*** *Suppose* $\forall n : |z_n| \leq M_n$ *with* $\sum_{j=0}^{\infty} M_j$ *convergent, then* $\sum_{j=0}^{\infty} z_j$ *converges.*
- *The series* $\sum_{j=0}^{\infty} c^j$ *converges iff* $|c| < 1$.
- ***Ratio Test:*** *Let* $L := \lim_{n\to\infty} |fracz_{n+1}z_n|$, *then* z_n *converges if* $L < 1$ *and diverges if* $L > 1$.
- ***Weierstrass M:*** *If* f_n *is a sequence of functions* $\forall j : |f_j| \leq M_j$ *and* $\sum_{j=0}^{\infty} M_j$ *converges then* f_n *converges uniformly.*

Definition. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, f_n **converges pointwise** to f if $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : |f_n(z) - f(z)| < \varepsilon$.

Definition. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions, f_n **converges uniformly** to f if $\forall\varepsilon > 0 : \exists N \in \mathbb{N} : \forall n \geq N : \forall z : |f_n(z) - f(z)| < \varepsilon$.

Theorem (4.1.21,4.1.22)**.** *If* f_n *converges uniformly we may commute limits with integrals (4.1.21) and integrals with sums (4.1.22).*

Theorem (4.1.23)**.** *If every* f_n *is holomorphic and* $f_n \to f$ *uniformly then* f *is holomorphic.*

Theorem (4.2.2)**.** *Let* $P = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ *be a power-series then* $\exists R \in [0,\infty]$: *(called the* ***radius of convregence****) such that:*

- P *converges on* $D_R(z_0)$,
- P *converges uniformly on* $D_r(z_0)$ *for any* $r < R$,
- P *diverges on* $\mathbb{C} \setminus \overline{D}_R(z_0)$.

Theorem (4.2.4)**.** *If* $r := \lim_{j\to\infty} \left|\frac{a_j}{a_{j+1}}\right|$ *exists then it's the radius of convergence of* $\sum_{j=0}^{\infty} a_j(z-z_0)^j$.

Theorem (4.2.6)**.** *The power series* $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ *with radius of convergence* R *is holomorphic on* $D_R(z_0)$.

Definition. The **Taylor seires** of f, for holomorphic f, is

$$\sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j.$$

Theorem (4.3.2)**.** *If* f *is holomorphic on* $D_R(z_0)$ *then it admits a Taylor* $\sum_{j=0}^{\infty} a_j(z-z_0)^j$ *series which converges uniformly with radius of convergence* R.

Definition. A function f is **analytic** if it admits a convergent power-series.

Theorem (4.3.5)**.** *Every holomorphic function is analytic.*

Theorem (4.3.9)**.** *The Taylor series of* $f'(z)$ *is the term-by-term derivative of the Taylor series of* $f(z)$ *(since Taylor series converge uniformly).*

Theorem (4.3.12)**.** *Taylor series are unique, specifically; the Taylor series of a function is equal to any valid power-series.*

Definition. The **Laurent series expansion** of a function f is $\sum_{j=-\infty}^{\infty} a_j(z-z_0)^j = \sum_{j=0}^{\infty} a_j(z-z_0)^j + \sum_{j=1}^{\infty} a_{-j}(z-z_0)^{-j}$.

Definition. The **open annulus of radii** r **and** R is $A_{r,R}(z_0) = D_R(z_0) \setminus D_r(z_0)$.

Theorem. *The coefficients of a Laurent series for a holomorphic function* f *are given by*

$$a_j = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{j+1}}dz,$$

for $\Gamma \in A_{r,R}(z_0)$.

Theorem (4.4.7)**.** *The Laurent series expansion of holomorphic* f *is unique.*

Definition. We say z_0 is a **singularity** of f if f isn't holomorphic at z_0. It is **isolated** if $\exists R > 0 : f$ is holomorphic on $D'(z_0)$, and of **order** m if $f(z_0) = \cdots = f^{(m-1)}(z_0) = 0 \neq f^{(m)}(z_0)$.

Theorem (4.5.5)**.** *If* $z_n \to z_0$ *and* $\forall n : z_n \in D$ *which is a neighbourhood domain of* z_0 *and* $f(z_0) = 0$ *then* $f(z) = 0$ *for all* $z \in D$.

Definition. Let z_0 be a singularity of a function f, then

- z_0 is **removable** if $\forall j < 0 : a_j = 0$,
- of **order** m if $\forall j < -m : a_j = 0$ but $a_j \neq 0$,
- **essential** if there are infinitely many $a_j \neq 0$ with $j < 0$.

Theorem (4.5.8)**.** *Let* $f = \sum_{j=0}^{\infty} a_j(z-z_0)^j$ *have removable singularity* z_0 *then re-defining* $f(z_0) = a_0$ *makes* f *holomorphic at* z_0.

Theorem (4.5.11)**.** *Let* f, g *be holomorphic at* z_0 *with* z_0 *a zero of order* m *of* g *then*

- *if* z_0 *isn't a zero of* f *then* f/g *has a pole of order* m *at* z_0,
- *if* z_0 *is a zero of order* k *of* f *then* f/g *has a pole of order* $m-k$ *at* z_0 *if* $m > k$ *and removable singularity otherwise.*

Definition. We say $F : \tilde{D} \to \mathbb{C}$ is an **analytic continuation** of $f : D \to \mathbb{C}$ with $\tilde{D} \subseteq D \subseteq \mathbb{C}$ if $F(z) = f(z)$ for $z \in D$ and F is holomorphic.

Theorem (4.6.4)**.** ***Identity theorem:*** *Let* $D \subseteq \mathbb{C}$ *be a domain with* f *holomorphic on* D *and* $f(z) = 0$ *for all* $z \in D_R(z_0) \subseteq D$ *then* $f(z) = 0$ *for all* $z \in D$.

Theorem (4.6.5)**.** *Let* $D \subseteq \mathbb{C}$ *be a domain with* $f, g : D \to \mathbb{C}$ *holomorphic with* $\forall z \in D_R(z_0) : f(z) = g(z)$ *then* $f(z) = g(z)$ *for all* $z \in D$.

Theorem (4.6.7)**.** *Let* $z_n \to z_0$ *and* $\forall n : f(z_n) = 0$ *with* $f : D \to \mathbb{C}$ *holomorphic, then* $\forall z \in D : f(z) = 0$.

Theorem (4.5.8)**.** *Let* $D \subseteq \mathbb{C}$ *be a domain with* $f, g : D \to \mathbb{C}$ *holomorphic with and* $z_n \to z_0, f(z_n) = g(z_n)$ *then* $\forall z \in D : f(z) = g(z)$ *(use this to prove that* $\sin^2(z) + \cos^2(z) = 1$ *holds for complex* $\sin, \cos$ *since* $f = g$ *on the real axis).*

NOTE:

$$\sum_{j=0}^{\infty} z^j \text{ is convergent on } D_1(0).$$

$$\sin(z) = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}$$

$$\cos(z) = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!}$$

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

Chapter Five - Residue calculus

Definition. The **residue** of a function f at isolated singularity z_0 is $\operatorname{Res}(f,z_0)=a_{-1}$, the coefficient of $\frac{1}{z-z_0}$ in the Laurent series expansion of f.

Theorem (5.1.4)**.** *Let f be holomorphic on $D'_R(z_0)$ with removable singularity z_0, then $\operatorname{Res}(f,z_0)=0$.*

Theorem (5.1.5)**.** *Let f be holomorphic on $D'_R(z_0)$ where z_0 is a pole of order m, then*

$$\operatorname{Res}(f,z_0)=\lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}((z-z_0)^m f(z)).$$

Theorem (5.1.7)**.** *Let g,h be holomorphic on $D'_R(z_0)$ where z_0 is a simple zero of h and $g(z_0)\neq 0$, then*

$$\operatorname{Res}(f,z_0)=\frac{g(z_0)}{h'(z_0)}.$$

Theorem. ***Cauchy Residue Theorem:*** *Let Γ be a loop with f holomorphic on $\operatorname{Int}(\Gamma)\setminus\{z_1,\dots,z_k\}$ for isolated singularities $z_1,\dots,z_k$. Then*

$$\int_\gamma f(z)dz=2\pi i\sum_{j=1}^{k}\operatorname{Res}(f,z_j).$$

Definition. A function f is **meromorphic** on a domain D if $\forall z\in D, f$ has a pole of finite order or is holomorphic.

Theorem. ***The Argument Principle:*** *Let Γ be a loop in $\mathbb{C}$ and f meromorphic on $\operatorname{Int}(\Gamma)$ and holomorphic on Γ, then*

$$\frac{1}{2\pi i}\int_\Gamma\frac{f'(z)}{f(z)}dz=N_0(f)-N_\infty(f),$$

where $N_0(f)=\sum_{j=1}^{l}\operatorname{ord}(w_j)$ is the sum of the orders of the zeros of f and $N_\infty(f)=\sum_{j=1}^{k}\operatorname{ord}(z_j)$ is the sum of the orders of the poles of f (the number of poles in $\operatorname{Int}(\Gamma)$, counted with multiplicity).

Theorem. ***Rouché's Theorem:*** *Let Γ be a loop and f,g be holomorphic inside and on Γ with*

$$\forall z\in\Gamma: |f(z)-g(z)|<|f(z)|$$

then $N_0(f)=N_0(g)$.

Theorem. ***Open-Mapping theorem:*** *Let $D\subseteq\mathbb{C}$ be a domain and f is non-constant and holomorphic on D, then the image $f(D)$ is open.*

Theorem. ***Maximum Modulus:*** *Let $D\subseteq\mathbb{C}$ be a domain and f be holomorphic and non-constant, then $|f(z)|$ doesn't attain its maximum on D.*

Theorem (5.2.18)**.** *Suppose f is holomorphic on domain D, if any of $\operatorname{Re}(f),\operatorname{Im}(f),|f|$, or $\operatorname{Arg}(f)$ are constant functions then f is also constant.*

Theorem. ***Jordan Lemma:*** *Let P/Q be rational with $\deg(Q)\geq\deg(P)+1$ then*

$$\lim_{\rho\to\infty}\int_C\exp(iaz)\frac{P(z)}{Q(z)}dz=0\quad\text{where } C=\begin{cases}C_\rho^+ & \text{for } a>0\\ C_\rho^- & \text{for } a<0\end{cases}$$

Theorem (5.5.3)**.** *Let D be a domain with f meromorphic on D with simple pole $c\in D$, if $\gamma:[\theta_0,\theta_1]\subseteq[0,2\pi]\to\mathbb{C},\theta\mapsto c+r\exp(i\theta)$ parametrizes the arc S_r then*

$$\lim_{r\to 0^+}\int_{S_r}f(z)dz=i(\theta_1-\theta_0)\operatorname{Res}(f,c).$$

. . . .

Sample questions

For **infinite series**:

$$\int_\Gamma\frac{\cot(\pi z)}{z^2}dz=\sum_n\operatorname{Res}(f,n)=\sum_{j=0}^{\infty}\frac{1}{r^2}=\frac{\pi}{6}.$$

For **finding series expansions**:

$$\frac{1}{1-f(z)}=\sum_{j=0}^{\infty}f(z)^j\quad\text{for } |f(z)|<1,$$

$$\frac{1}{(1-z)^2}=\frac{d}{dz}\frac{1}{1-z}$$

$$\frac{1}{z-k}=\frac{1}{z}\cdot\frac{1}{1-k/z}=\sum_{j=0}^{\infty}\left(\frac{k}{z}\right)^j.$$

For **maps to the unit disc:** The function

$$f(z)=\frac{z-1}{z+1},$$

maps the imaginary axis (equation $|z-1|=|z+1|$) to the unit circle.

On **any disc**: On $z\in D_r(z_0)$ notice that $z\bar{z}=|z|^2\implies\bar{z}=\frac{r^2}{z}$ and so

$$\operatorname{Re}(z)=\frac{z+\bar{z}}{2}=\frac{z+r^2/z}{2}.$$

Workshop 4; Question 6: The function $f(z)=(z^2+1)^{1/2}$ has branches given by $\exp(\frac{1}{2}g(z))$ for $g(z)\in\log(z^2+1)=\log(z+i)+\log(z-i)$ a branch og the logarithm. Thus the branch of f holomorhpic on the unit disc $D_1(0)$ is given by:

$$f(z)=\exp\left(\frac{1}{2}\left(\operatorname{Log}_{-\pi/2}(z+i)+\operatorname{Log}_{\pi/2}(z-i)\right)\right).$$

Workshop 5; Question 2:

$$f_1(z)=\frac{z-i}{z+i}:\qquad U_1=\{z\in\mathbb{C}:\operatorname{Im}(z)>0\}\to D_1(0),$$

$$f_2(z)=\frac{\exp(\pi z)-i}{\exp(\pi z)+i}:\quad U_2=\{z\in\mathbb{C}:0<\operatorname{Im}(z)<1\}\to D_1(0),$$

$$f_3(z)=\frac{z^2-i}{z^2+i}:\qquad U_3=\{z\in\mathbb{C}:\operatorname{Im}(z),\operatorname{Re}(z)>0\}\to D_1(0),$$

$$f_4(z)=\frac{z^2-1}{z^2+1}:\qquad U_4=\left\{z\in\mathbb{C}:\operatorname{Arg}(z)\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)\right\}\to D_1(0).$$

Workshop 6; Question 3:

$$\mathbb{R}\ni\left|\int_\Gamma f(z)dz\right|\neq\int_\Gamma|f(z)|dz\in\mathbb{C}$$

Useful Formulae

$$\sin(a\pm b)=\sin(a)\cos(b)\pm\cos(a)\sin(b)$$
$$\cos(a\pm b)=\cos(a)\cos(b)\mp\sin(a)\sin(b)$$
$$\sin(z)=\frac{\exp(iz)-\exp(-iz)}{2i}$$
$$\cos(z)=\frac{\exp(iz)+\exp(-iz)}{2}$$

$$\sin(iz)=i\sinh(z)$$
$$\cos(iz)=\cosh(z)$$
$$\cosh^2(z)-\sinh^2(z)=1$$

Trigonometric Integral example

Theorem.

$$\int_0^{2\pi} R(\cos\theta,\sin\theta)d\theta=\int_\Gamma f(z)dz,\ where\ f(z)=\frac{R(\cos\theta,\sin\theta)}{i\exp(i\theta)}.$$

Example. Consider the integral

$$I=\int_0^{2\pi}\frac{\sin^2\theta}{5+4\cos\theta}d\theta.$$

Noting that $z=e^{i\theta}=\cos\theta+i\sin\theta$ and so $\cos\theta=\mathrm{Re}(e^{i\theta})=\frac{z+\bar{z}}{2}$ which on the unit circle becomes $\cos\theta=\frac{z+\frac{1}{z}}{2}$ (since $1=|z|^2=z\bar{z}$), then:

$$f(z)=\frac{1}{iz}\frac{\left(\frac{1}{2i}\left(z-\frac{1}{z}\right)\right)^2}{5+4\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)}=\frac{i}{8}\frac{(z^2-1)^2}{z^2(z+\frac{1}{2})(z+2)},$$

with poles at $-\frac{1}{2},-2$ and 0, of which only 0 and $-\frac{1}{2}$ lie inside $D_1(0)$ and so we calculate their residues:

$$\mathrm{Res}(f,0)=\lim_{z\to 0}\frac{d}{dz}\left(\frac{i}{8}\frac{(z^2-1)^2}{(z+\frac{1}{2})(z+2)}\right)=\frac{-5i}{16}$$
$$\mathrm{Res}(f,1/2)=\lim_{z\to\frac{1}{2}}\frac{d}{dz}\left(\frac{i}{8}\frac{(z^2-1)^2}{z^2(z+2)}\right)=\frac{3i}{16}$$

giving that

$$I=2\pi i\sum_{z_j\text{ is a pole}}z_j=2\pi i\left(\frac{-5i}{16}+\frac{3i}{16}\right)=\frac{\pi}{4}.$$