Honours Complex Variables

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Holomorphic Functions

Lemma 1.1.14.
Let $z, w \in \mathbb{C}$, then

(i) $|z| = 0 \Leftrightarrow z = 0$;
(ii) $|\overline{z}| = |z|$;
(iii) $|zw| = |z||w|$;
(iv) $\overline{\overline{z}} = z$;
(v) $|z|^2 = z\overline{z}$
(vi) $\overline{z+w} = \overline{z} + \overline{w}$;
(vii) $\overline{zw} = (\overline{z})(\overline{w})$;
(viii) $|\operatorname{Re}(z)| \leqslant |z|$ and $|\operatorname{Im}(z)| \leqslant |z|$;
(ix) $\operatorname{Re}(z) = \dfrac{z+\overline{z}}{2}$ and $\operatorname{Im}(z) = \dfrac{z-\overline{z}}{2i}$

Remark (unknown).
Let $z \in \mathbb{C}$. If $|z| = 1$, then $\overline{z} = \dfrac{1}{z}$.

Lemma 1.1.15 (Triangle Inequality).
Let $z, w \in \mathbb{C}$, then

$$|z+w| \leqslant |z| + |w|$$

Lemma 1.1.16 (Reverse Triangle Inequality).
Let $z, w \in \mathbb{C}$, then

$$|z-w| \geqslant ||z| - |w||$$

Proposition 1.1.19.
Let $z, w \in \mathbb{C} \setminus \{0\}$. Then

(i) $\arg(zw) = \arg(z) + \arg(w)$ and $\arg(\overline{z}) = -\arg(z)$;
(ii) $\operatorname{Arg}(zw) = \operatorname{Arg}(z) + Arg(w) + 2k\pi$ and $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z) + 2m\pi$, $k, m \in \mathbb{Z}$.

Theorem 1.4.5 (Cauchy-Riemann Equations).
Let $z_0 = z_0 + iy_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ a neighbourhood of z_0 and $f : U \to \mathbb{C}$ differentiable at z_0, where $f = u + iv$. Then

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0),$$
$$\frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0)$$

Theorem 1.4.6.
Let $z_0 = z_0 + iy_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ a neighbourhood of z_0 and $f : U \to \mathbb{C}$ with $f = u + iv$. If u, v are ***continuously differentiable***, i.e. derivatives exist and are continuous, on a neighbourhood of (x_0, y_0) and satisfy the Cauchy-Riemann equations at (x_0, y_0), then f is differentiable at z_0.

Example 1.4.11.
$|z|^2$ is differentiable only at the origin and ***nowhere*** holomorphic.

Lemma 1.4.13.
Let $u, v : \mathbb{R}^2 \to \mathbb{R}$ be twice continuously differentiable, i.e. all second partial derivatives exist and are continuous. If $f(x+iy) = u(x,y) + iv(x,y)$ is holomorphic on $\mathbb{C}$, then u, v are harmonic.

Lemma 1.5.6.
Let $P, Q : \mathbb{C} \to \mathbb{C}$ be polynomials. Then rational function $R = P/Q$ is holomorphic on $\{z \in \mathbb{C} : Q(z) \neq 0\}$.

Lemma 1.6.6.
Let $z, w \in \mathbb{C}$. Then

(i) $\sin(z + \pi/2) = \cos(z)$;
(ii) $\sin(z+w) = \sin(z)\cos(w) + \cos(z)\sin(w)$;
(iii) $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$

Lemma 1.6.7.
Let $z = x + iy \in \mathbb{C}$. Then

$$\sin(x+iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y),$$
$$\cos(x+iy) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$

Lemma 1.6.10.

$$\sinh(iz) = i\sin(z), \quad \cosh(iz) = \cos(z)$$

Lemma 1.7.3.
Let $z, w \in \mathbb{C} \setminus \{0\}$. Then

(i) $\log(z) = \ln|z| + i\arg(z) = \{\ln|z| + i\operatorname{Arg}(z) + 2\pi ik : k \in \mathbb{Z}\}$;
(ii) $\log(zw) = \log(z) + \log(w)$;
(iii) $\log(1/z) = -\log(z)$

Lemma 1.8.2.
We can rewrite z^α as:

$$z^\alpha = \{\exp(\alpha\ln|z| + i\alpha\operatorname{Arg}(z) + i\alpha 2\pi k) : k \in \mathbb{Z}\}$$
$$= \{\exp(\alpha\operatorname{Log}(z))\exp(i\alpha 2\pi k) : k \in \mathbb{Z}\}$$

Theorem 1.8.3.
Let $\alpha, z \in \mathbb{C}$, $z \neq 0$. Then

(i) $\alpha \in \mathbb{Z} \Rightarrow$ one value of z^α;
(ii) $\alpha = p/q$, with p, q coprime integers, $q \neq 0$ $\Rightarrow$ exactly q values of z^α;
(iii) α irrational or non-real $\Rightarrow$ infinitely many values of z^α.

Lemma 1.8.8.
Let $\alpha, \beta, z \in \mathbb{C}$, with $z \neq 0$. Then $z^\alpha z^\beta = z^{\alpha+\beta}$, where principal branch of logarithm is chosen for each power.

Exercise 1.8.11.
Let $z, w, \alpha \in \mathbb{C}$. It is not true in general that $(zw)^\alpha = z^\alpha w^\alpha$, where principal branch is chosen in each case. Consider

Remark (TopHat).
Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then f maps bounded sets to bounded sets.

Remark (TopHat).
Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then f ***does not*** map unbounded sets to unbounded sets, consider $f(z) = a_0 \in \mathbb{C}$, i.e. $f(\mathbb{C}) = \{a_0\}$.

Question Ws.1, Q.7.
Let $z \in \mathbb{C}$, then

$$|z| \leqslant |\operatorname{Re}(z)| + |\operatorname{Im}(z)| \leqslant \sqrt{2}|z|$$

Question Ws.2, Q.1 (De Moivre's Formula).
Let $\theta \in \mathbb{R}, n \in \mathbb{Z}$. Then:

$$\cos(n\theta) + i\sin(n\theta) = (\cos\theta + i\sin\theta)^n$$

Question Ws.2, Q2.
(b) Let $z \in \mathbb{C} \setminus \{0\}$, then $\arg(z^2) \neq 2\arg(z)$ in general, e.g. $z = -1$.

Question Ws.2, Q.3.
(b) Let $z \in \mathbb{C} \setminus \{0\}$, then $\arg(1/z) = \arg(\overline{z}) = -\arg(z)$.

Question Ws.2, Q.6.
Let $z \in \mathbb{C}$ and $z \neq 1$, then

$$\sum_{k=0}^{m} z^k = \frac{1 - z^{m+1}}{1-z}$$

Question Ws.3, Q.1.
$f(z) = |z|$ is continuous everywhere on $\mathbb{C}$, but nowhere holomorphic.

Question Ws.3, Q.6.
Let f be real-valued and holomorphic. Then f is constant.

Question Ws.3, Q.7.
Let $f : \mathbb{C} \to \mathbb{C}$ be holomorphic. Then

(a) $\overline{f(\overline{z})}$ is holomorphic;
(b) if $\overline{f(z)}$ is holomorphic, f is constant;
(c) if $f(\overline{z})$ is holomorphic, f is constant.

Conformal Maps and Möbius Transformations

Theorem 2.1.2.
Let $U \subseteq \mathbb{C}$ be open and $f : U \to \mathbb{C}$ holomorphic. Then f preserves angles at every $z_0 \in U$ where $f'(z_0) \neq 0$.

Remark 2.2.2.
If f is a Möbius transformation defined by $a, b, c, d \in \mathbb{C}$ and $\lambda \in \mathbb{C}$, then $\lambda a, \lambda b, \lambda c, \lambda d$ define the same Möbius transformation:

$$\frac{\lambda az + \lambda b}{\lambda cz + \lambda d} = \frac{az+b}{cz+d}$$

i.e. we can impose condition $ad - bc = 1$.

Lemma 2.2.3.
Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with determinant $ad - bc = 1$, then we associate the Möbius transformation $f_M(z) = \frac{az+b}{cz+d}$. Under this correspondence:

$$f_{M_1 M_2} = f_{M_1} \circ f_{M_2}, \quad f_{M^{-1}} = f_M^{-1}$$

Theorem 2.4.2.
Let f be a Möbius transformation. Then f is a composition of a ***finite*** number of translations, rotations, dilations and if and only if f does ***not*** fix the point at infinity, one inversion.

Corollary 2.4.3.
Möbius transformations map circlines to circlines.

Lemma 2.5.1.
Let f be a Möbius transformation and $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ three ***distinct*** points s.t. $f(z_2) = z_2$, $f(z_3) = z_3$, $f(z_4) = z_4$. Then f is the identity.

Theorem 2.5.2.
Let $z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be three ***distinct*** points. Then there exists a ***unique*** Möbius transformation s.t. $f(z_2) = 1$, $f(z_3) = 0$, $f(z_4) = \infty$.

Corollary 2.5.3.
Let $(z_2, z_3, z_4), (w_2, w_3, w_4) \in \tilde{\mathbb{C}}$ be two triplets of ***distinct*** points. Then there exists a ***unique*** Möbius transformation f s.t. $f(z_2) = w_2$, $f(z_3) = w_3$, $f(z_4) = w_4$.

Remark 2.5.6.
Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$, then:

$$[z_1, z_2, z_3, z_4] = \frac{z_1 - z_3}{z_1 - z_4}\frac{z_2 - z_4}{z_2 - z_3}$$

If one of the z_i is ∞, then all terms involving it disappear, e.g.:

$$[z_1, z_2, \infty, z_4] = \frac{z_2 - z_4}{z_1 - z_4}$$

Theorem 2.5.7.
Let $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be distinct and f a Möbius transformation. Then

$$[f(z_1), f(z_2), f(z_3), f(z_4)] = [z_1, z_2, z_3, z_4]$$

Question Ws.5, Q.1.

(a) $f(z) = \dfrac{(z-1)}{(z+1)}$:

  (i) $f(\{\operatorname{Re}(z) > 0\}) = D_1(0)$
  (ii) $f(D_1(0)) = \{\operatorname{Re}(z) < 0\}$

(b) $f(z) = \exp(iz)$:

(i) $f(\{0 < \mathrm{Re}(z) < \pi\}) = \{\mathrm{Im}(z) > 0\}$

(ii) $f(\{-\pi/2 < \mathrm{Re}(z) < \pi/2 \text{ and } \mathrm{Im}(z) > 0\}) = \{|z|, -\pi/2 < \mathrm{Arg}(z) < \pi/2\}$

(c) $f(z) = z^{\frac{1}{2}}$:

(i) $f(\{\mathrm{Re}(z) > 0\}) = \{-\pi/4 < \mathrm{Arg}(z) < \pi/4\}$

(ii) $f(D_{0,-\pi}) = \{\mathrm{Re}(z) > 0\}$ (preimage is cut plane)

Complex Integration

Lemma 3.2.8.
Let Γ be arc of a circle of radius r traced through angle θ. Then $\ell(\Gamma) = r\theta$.

Lemma 3.2.9 (M-L Lemma).
Let $\Gamma \in \mathbb{C}$ be a regular curve and let $f : \Gamma \to \mathbb{C}$ be ***continuous***. Then

$$\left| \int_\Gamma f(z)\, dz \right| \leqslant \ell(\Gamma) \max_{z \in \Gamma} f(z)$$

Lemma 3.3.2.
Let $D \subseteq \mathbb{C}$ be a domain an suppose $u : D \to \mathbb{R}$ is differentiable and $\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial y}$ on D. Then u is constant on D.

Theorem 3.3.5 (Fundamental Theorem of Calculus).
Let $D \subseteq \mathbb{C}$ be a domain, $\Gamma \subseteq D$ contour joining $z_0, z_1 \in D$, $f : D \to \mathbb{C}$ with antiderivative F on D. Then

$$\int_\Gamma f(z)\, dz = F(z_1) - F(z_0)$$

Corollary 3.3.6.
Let $D \subseteq \mathbb{C}$ be a domain, f holomorphic on D with $\forall z \in D : f'(z) = 0$. Then f is constant.

Lemma 3.3.9 (Path-Independence Lemma).
Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ ***continuous***. Then the following are equivalent:

(i) f has an antiderivative on D;

(ii) $\int_\Gamma f(z)\, dz = 0$ for all closed contours Γ on D;

(iii) all $\int_\Gamma f(z)\, dz$ are independent of path.

Theorem 3.4.2 (Jordan Curve Theorem).
Let $\Gamma \subseteq \mathbb{C}$ be a loop. Then Γ defines two regions, bounded domain $\mathrm{Int}(\Gamma)$ and unbounded domain $\mathrm{Ext}(\Gamma)$, with common boundary Γ.

Theorem 3.4.8 (Cauchy Integral Theorem).
Let f holomorphic inside and on loop Γ. Then

$$\int_\Gamma f(z)\, dz = 0$$

Corollary 3.4.9.
Let $D \subseteq \mathbb{C}$ be a ***simply-connected*** domain, f holomorphic on D. Then f has antiderivative on D.

Remark (unknown).
Due to Cauchy Integral Theorem, we can deform a contour without changing value of integral, provided we do not cross any point where f is not holomorphic.

Theorem 3.4.11.
Let $z_0 \in \mathbb{C}$ and $\Gamma \subseteq \mathbb{C}$ a loop s.t. it does not pass through z_0. Then

$$\int_\Gamma \frac{1}{z - z_0} = \begin{cases} 2\pi i & z_0 \in \mathrm{Int}(\Gamma), \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.5.1 (Cauchy Integral Formula).
Let Γ be a loop, $z_0 \in \mathrm{Int}\,\Gamma$ and f holomorphic inside and on Γ. Then

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0}\, dz.$$

Corollary 3.5.4.
Let $D \subseteq \mathbb{C}$ be a domain and f holomorphic on D. Then f is infinitely differentiable on D and all derivatives are holomorphic on D.

Theorem 3.5.5 (Generalized Cauchy Integral Formula).
Let Γ be a loop, $z_0 \in \mathrm{Int}\,\Gamma$ and f holomorphic inside and on Γ. Then f is infinitely differentiable at z_0 and $\forall n \in \mathbb{N}$:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Theorem 3.5.11 (Morera's Theorem).
Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$ ***continuous*** s.t. $\int_\Gamma f(z)\, dz = 0$ for all loops $\Gamma \subseteq D$. Then f is holomorphic.
Hint: Antiderivative by Path-Independence & Corollary 3.5.4.

Theorem 3.6.1.
Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ s.t. $\overline{D}_R(z_0) \subseteq D$, f holomorphic on D and $M > 0$ s.t. $\forall z \in D : |f(z)| \leqslant M$. Then $\forall n \in \mathbb{N}$: $|f^{(n)}(z_0)| \leqslant \dfrac{n!M}{R^n}$.
Hint: Generalized Cauchy Integral Formula and Lemma 3.2.9.

Theorem 3.6.2 (Liouville's Theorem).
Let f be holomorphic on $\mathbb{C}$ and bounded, i.e. there exists $M > 0$ s.t. $\forall z \in \mathbb{C} |f(z)| \leqslant M$. Then f is constant.
Hint: Theorem 3.6.1 on circle $\Rightarrow f'(z) = 0 \Rightarrow f$ constant by Corollary 3.3.6.

Exercise 3.6.4.
Let P be a (monic) polynomial of degree N, then there exists $R > 0$ s.t. $|z| \geqslant R \Rightarrow |P(z)| \geqslant \frac{1}{2}|z|^N$.

Theorem 3.7.1.
Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ s.t. $\overline{D}_R z_0 \subseteq D$ and f holomorphic on D. Then

$$f(z_0) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + Re^{it})\, dt$$

Remark 3.7.2.
If there exists $M > 0$ s.t. $\forall z \in C_R(z_0) : |f(z)|$ with requirements of Theorem 3.7.1, then $|f(z_0)| \leqslant M$.

Lemma 3.7.3.
Let $D \subseteq \mathbb{C}$ be a domain, $z_0 \in D$ and $R > 0$ s.t. $\overline{D}_R(z_0) \subseteq D$, f holomorphic on D s.t. $\max_{z \in \overline{D}_R(z_0)} |f(z)| = |f(z_0)|$. Then $|f(z)|$ is constant on $\overline{D}_R(z_0)$.

Exercise 3.7.4.
Let $D \subseteq \mathbb{C}$ be a domain, f holomorphic on D s.t. $|f(z)|$ is constant on D. Then f is constant on D.

Theorem 3.7.5 (Maximum Modulus Principle).
Let $D \subseteq \mathbb{C}$ be a domain, f holomorphic ***and bounded*** on D, i.e. $|f(z)| \leqslant M$ for $M > 0$. If $|f(z)|$ achieves maximum at $z_0 \in D$, then f is constant.

Remark 3.7.6.
A holomorphic function on a bounded domain, continuous up to and including the boundary, attains maximum on the boundary.

Theorem 3.7.8 (Maximum/minimum Principle for Harmonic Functions).
Let $D \subseteq \mathbb{R}^2$ be a domain, $\phi : D \to \mathbb{R}$ be harmonic s.t. ϕ is bounded above or below on D by $M > 0$ and $\exists z_0 \in D : \phi(z_0) = M$. Then ϕ is constant on D.

Question Ws.7, Q.5.
Let f be holomorphic on $D_1(0)$ s.t. $\max_{z \in C_r(0)} |f(z)| \to 0$ as $r \to 1$, then $f = 0$.

Question Ws.8, Q.2.
Let f be holomorphic on $\mathbb{C}$ s.t. $|f| \to 0$ as $|z| \to \infty$. Then $\forall z \in \mathbb{C} : f(z) = 0$.

Question Ws.8, Q.3.
Let f be holomorphic on $\mathbb{C}$ and periodic in real and imaginary directions, i.e. $\exists a_0, b_0 > 0 \forall z \in \mathbb{C} : f(z) = f(z + z_0)$ and $f(z) = f(z + ib_0)$. Then f is constant.
Hint: f is determined by values within rectangle, so bounded. Then Liouville's Theorem.

Question Ws.8, Q.4.
Let f be holomorphic on $\mathbb{C}$. If $\mathrm{Re}(f(z))$ ***or*** $\mathrm{Im}(f(z))$ are bound below ***or*** above for all $z \in \mathbb{C}$, then f is constant.

Question Ws.8, Q.5.
Let f be holomorphic on $\mathbb{C}$ s.t. for some integer $N \geqslant 1$ there exists $C > 0$ s.t. $|f(z)| \leqslant C|z|^N$ for all $z \in \mathbb{C}$. Then $f^(n)(z) = 0$ for all $z \in \mathbb{C}$, for all $n \geqslant N + 1$.

Question Ws.8, Q.6.
Suppose f is holomorphic on $\mathbb{C}$ s.t. $|f(z)| \to \infty$ as $|z| \to \infty$. Then f is surjective.

Question Ws.9, Q.4.
Let f be holomorphic on $\mathbb{C}$ s.t. there exists $C > 0$ s.t. $|f(z)| \leqslant C|z|^2$ for all $z \in \mathbb{C}$. Then $f(z) = cz^2$ for some $c \in \mathbb{C}$ s.t. $|c| \leqslant C$.

Infinite Series

Lemma 4.1.2.
Let $\sum_{j=0}^\infty z_j$ be a convergent series. Then $z_j \to 0$ as $j \to \infty$.

Lemma 4.1.6 (Comparison Test).
Let $z_n \in \mathbb{C}$ be a sequence s.t. $|z_n| \leq M_n$, for $M_n \geq 0$, for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$, where $\sum_{j=0}^\infty M_j$ is convergent. Then $\sum_{j=0}^\infty z_j$ is convergent.

Lemma 4.1.7.
Let $c \in \mathbb{C}$. Then $\sum_{j=0}^\infty c^j$ is convergent if and only if $|c| < 1$.

Lemma 4.1.9 (Ratio Test).
Let $z_n \in \mathbb{C}$ be a sequence and suppose

$$\lim_{n \to \infty} \left| \frac{z_{n+1}}{z_n} \right| = L$$

Then

(i) if $L < 1$, the series $\sum_{j=0}^\infty z_j$ is convergent;

(ii) if $L > 1$, the series $\sum_{j=0}^\infty z_j$ is divergent;

(iii) if $L = 1$, we can conclude nothing.

Example 4.1.15.
Let $f_n(z) = \exp(-nz^2)$ (***holomorphic***!), then $f_n \to f$ as $n \to \infty$ ***pointwise*** where

$$f(x) = \begin{cases} 1 & x = 0, \\ 0 & x \neq 0. \end{cases}$$

is ***not*** holomorphic.

Lemma 4.1.17.
Let $S \subseteq \mathbb{C}$ and suppose $f_n : S \to \mathbb{C}$, sequence of continuous functions, converge ***uniformly*** to f. Then f is continuous.

Lemma 4.1.19 (Weierstrass M-test).
Let $S \subseteq \mathbb{C}$, $f_n : S \to \mathbb{C}$ a sequence of functions and $M_n \geq 0$ a sequence of non-negative numbers s.t. for all $z \in S$ and for all $n \geq n_0 \in \mathbb{N}$, $|f_n(z) \leq M_n|$ and $\sum_{j=0}^{\infty} M_j$ converges. Then $\sum_{j=0}^{\infty} f_j(z)$ converges uniformly on S.

Theorem 4.1.23.
Let $D \subseteq \mathbb{C}$ be a ***simply-connected*** domain, f_n holomorphic on D and converge uniformly to f. Then $f : D \to \mathbb{C}$ is holomorphic on D.

Theorem 4.2.4 (
). Let $\sum_{j=0}^{\infty} a_j(z - z_0)^j$ be a power series and suppose the sequence $\left|\frac{a_j}{a_{j+1}}\right|$ has a limit. Then the radius of convergence is equal to this limit.

Exercise 4.3.8.
The following Taylor series are centred at 0:

$$\exp(z) = \sum_{j=0}^{\infty} \frac{z^j}{j!}$$

$$\cos(z) = \sum_{j-0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!}$$

$$\sin(z) = \sum_{j-0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!}$$

Theorem 4.4.4 (Laurent Series).
Let $z_0 \in \mathbb{C}$, $0 \leqslant r < R \leqslant \infty$, f holomorphic on $A_{r,R}(z_0)$. Then f can be expressed as Laurent series centred at z_0, convergent on $A_{r,R}(z_0)$ and ***uniformly*** convergent on $\overline{A}_{r_1,R_1}(z_0)$ for $r < r_1 \leqslant R_1 < R$. Moreover:

$$a_j = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - z_0)^{j+1}}\, dz$$

Proposition 4.5.4.
Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ a neighbourhood of z_0, f holomorphic on U with zero of finite order z_0. Then z_0 is isolated.
Hint: Function with Zeros Trick, $g(z_0) \neq 0$ and continuity of g.

Corollary (Lecture).
Let f have finitely many zeros. Then all zeros are isolated.

Corollary 4.5.5.
Let $z_0 \in \mathbb{C}$, $U \subseteq \mathbb{C}$ a neighbourhood of z_0, f holomorphic on U s.t. $f(z_n) = 0$ for sequence $z_n \in U$ s.t. $z_n \to z_0$ as $n \to \infty$. Then $\exists R > 0$ s.t. $\forall z \in D_R(z_0) : f(z) = 0$.
Hint: Continuity of f and contrapositive of Prop. 4.5.4.

Corollary 4.5.6.
Let $z_0 \in \mathbb{C}$ be singularity of rational function $f = P/Q$. Then z_0 is isolated.

Theorem 4.5.8.
Let $z_0 \in \mathbb{C}$ be a removable singularity of f, holomorphic on $D'_R(z_0)$ for some $R > 0$. Then $f(z_0)$ can be (re-)defined s.t. f is holomorphic on z_0.

Lemma 4.5.11.
Let f, g be holomorphic at z_0, where z_0 is zero of order m of g. Then

(i) if z_0 is not zero of f, f/g has pole of order m at z_0;
(ii) if z_0 is zero of order k of f, f/g has pole of order $m - k$ at z_0 if $m > k$ and removable singularity otherwise.

Hint: Function with Zeros Trick.

Theorem 4.6.4 (Identity Theorem).
Let $D \subseteq \mathbb{C}$, $z_0 \in D$, f holomorphic on D s.t. $\forall z \in D_R(z_0) : f(z) = 0$ for some $R > 0$. Then $f(z) = 0$ for all $z \in D$.

Corollary 4.6.5.
Let $D \subseteq \mathbb{C}$, f, g holomorphic on D s.t. $\forall z \in D_R(z_0) : f(z) = g(z)$ for some $R > 0$. Then $f(z) = g(z)$ for all $z \in D$.

Corollary 4.6.7.
Let $D \subseteq \mathbb{C}$, $z_0 \in D$ and f holomorphic on D s.t. $f(z_n) = 0$ for a sequence of distinct $z_n \in D$ which converge to z_0. Then $f(z) = 0$ for all $z \in D$.

Corollary 4.6.8.
Let $D \subseteq \mathbb{C}$, $z_0 \in D$ and f, g holomorphic on D s.t. $f(z_n) = g(z_n)$ for a sequence of distinct $z_n \in D$ which converge to z_0. Then $f(z) = g(z)$ for all $z \in D$.

Question Ws.10, Q.5.
Let f be holomorphic on $D'_r(z_0)$ and $|f(z)| \leqslant M$ for all $z \in D'_r(z_0)$, for some $M > 0$. Then f can be (re-)defined at z_0 to make f holomorphic on $D_r(z_0)$.

Residue Calculus

Theorem 5.1.1.
Let $z_0 \in \mathbb{C}$, f holomorphic on $D'_R(z_0)$ for some $R > 0$ with z_0 being isolated singularity, Γ in $D'_R(z_0)$ and $z_0 \in \mathrm{Int}(\Gamma)$. Then

$$\int_{\Gamma} f(z)\, dz = 2\pi i a_{-1}$$

where a_{-1} is coefficient from Laurent expansion.

Lemma 5.1.4.
Let $z_0 \in \mathbb{C}$, f holomorphic on $D'_R(z_0)$ for some $R > 0$, with ***removable*** singularity z_0. Then $\mathrm{Res}(f, z_0) = 0$.

Lemma 5.1.5.
Let $z_0 \in \mathbb{C}$, f holomorphic on $D'_R(z_0)$ for some $R > 0$, with pole of order m at z_0. Then
$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)]$.
Hint:

Lemma 5.1.7.
Let $z_0 \in \mathbb{C}$, g and h holomorphic on $D'_R(z_0)$ for some $R > 0$, s.t. h has a ***simple*** zero at z_0, while $g(z_0) \neq 0$. Then for $f = g/h$:

$$\mathrm{Res}(f, z_0) = \frac{g(z_0)}{h'(z_0)}.$$

Theorem 5.1.11 (Cauchy Residue Theorem).
Let f be holomorphic inside and on loop Γ except for ***finitely*** many isolated singularities $z_1, \ldots, z_k \in \mathrm{Int}(\Gamma)$. Then

$$\int_{\Gamma} f(z)\, dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j).$$

Theorem 5.2.5 (The Argument Principle).
Let $\Gamma \subseteq \mathbb{C}$ be a loop, f ***non-zero on*** Γ, holomorphic inside and on Γ, except for ***finitely*** many poles in Γ (meromorphic). Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = \sum_{z_0 \in \mathrm{Int}(\Gamma)} \mathrm{order}(z_0) - \sum_{z_\infty \in \mathrm{Int}(\Gamma)} \mathrm{order}(z_\infty)$$

where z_0 is a zero of f and z_∞ is a pole of f.

Corollary 5.2.6.
Let $\Gamma \subseteq \mathbb{C}$ be a loop, f ***non-zero on*** Γ, holomorphic inside and on Γ. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)}\, dz = \sum_{z_0 \in \mathrm{Int}(\Gamma)} \mathrm{order}(z_0)$$

where z_0 is a zero of f.

Theorem 5.2.7 (Rouché's Theorem).
Let Γ be a loop, f, g holomorphic inside and on Γ s.t. $\forall z \in \Gamma : |f(z) - g(z)| < |f(z)|$. Then

$$\sum_{z_0 \in \mathrm{Int}(\Gamma)} \mathrm{order}(z_0) = \sum_{z_0 \in \mathrm{Int}(\Gamma)} \mathrm{order}(w_0)$$

where z_0 is zero of f and w_0 is zero of g. N.B.: Number and order of zeros can be different, only total is equal.

Theorem 5.2.16 (Open Mapping Theorem).
Let $D \subseteq \mathbb{C}$ be a domain and suppose $f : D \to \mathbb{C}$ is non-constant and holomorphic on D. Then $f(D)$ is an open subset of $\mathbb{C}$.

Corollary 5.2.18.
Let $D \subseteq \mathbb{C}$ be a domain, f holomorphic on D s.t. ***any*** of the values $\mathrm{Re}(f(z))$, $\mathrm{Im}(f(z))$, $|f(z)|$, or $\mathrm{Arg}(f(z))$ is constant. Then f is constant.

Exercise 5.2.19 (Schwarz's Lemma).
Let f be holomorphic on $D_1(0)$ s.t. $f(0) = 0$ and $\forall z \in D_1(0) : |f(z)| \leqslant 1$. Then $|f(z)| \leqslant |z|$.

Remark (unknown).
Let $z = e^{i\theta}$, i.e. $z \in D_1(0)$, then

$$\cos\theta = \mathrm{Re}(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$\sin\theta = \mathrm{Im}(z) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

Remark (Trigonometric Integrals).
Let $\Gamma = C_1(0)$, parametrized by $\gamma : [0, 2\pi] \to \mathbb{C}$; $\theta \mapsto \exp(i\theta)$ and let $R(\cos\theta, \sin\theta)$ be a rational function of cosines and sines, then

$$\int\limits_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta = \int_{\Gamma} \frac{1}{iz} R\left(\frac{z + \frac{1}{z}}{2}, \frac{z - \frac{1}{z}}{2i}\right) dz$$

i.e. trigonometric integral can be evaluated as contour integral on unit circle by replacing $\cos\theta$ with $\frac{z+1/z}{2}$ and $\sin\theta$ with $\frac{z-1/z}{2i}$ and multiplying integrand by $\frac{1}{iz}$.

Lemma 5.4.6 (Jordan Lemma).
Let P, Q be polynomials s.t. $\deg(Q) \geqslant \deg(P) + 1$ and $a \in \mathbb{R} \setminus \{0\}$. Then

$$\lim_{\rho \to \infty} \int_{C_\rho^+} \exp(iaz) \frac{P(z)}{Q(z)}\, dz = 0, \quad \text{if } a > 0;$$

$$\lim_{\rho \to \infty} \int_{C_\rho^-} \exp(iaz) \frac{P(z)}{Q(z)}\, dz = 0, \quad \text{if } a < 0$$

where C_ρ^+, C_ρ^- are semicircular arcs from $-\rho$ to ρ in upper/lower half-plane.

Lemma 5.5.3.
Let $D \subseteq \mathbb{C}$ be a domain, $c \in D$, f holomorphic on D except at possibly finitely many

singularities with ***simple*** pole at c. Let S_r be circular arc parametrized by $\gamma(\theta) = c + r\exp(i\theta)$ for $\theta \in [\theta_0, \theta_1]$ for some $0 \leqslant \theta_0 < \theta_1 \leqslant 2\pi$. Then

$$\lim_{r \to 0} \int_{S_r} f(z)\, dz = i(\theta_1 - \theta_0)\mathrm{Res}(f, c)$$

Lemma 5.6.4.
Let $0 \leqslant k \leqslant n$ be non-negative integers, let $\binom{n}{k}$ be the usual binomial coefficient and Γ a loop with 0 in interior. Then

$$\binom{n}{k} = \frac{1}{2\pi i} \int_\Gamma \frac{(1+z)^n}{z^{k+1}}\, dz$$

Miscellaneous

Example (Circle Parametrization).
Circle $C_r(z_0)$, $z_0 \in \mathbb{C}, r > 0$ can be parametrized by
$\gamma : [0, 2\pi] \to \mathbb{C}; t \mapsto z_0 + r\exp(it)$.

Remark (Contour Integral Checklist).

- ☐ Accounted for orientation of contour?
- ☐ Accounted for $\frac{n!}{2\pi i}$ factor in (Generalized) Cauchy Integral Formula?

Remark (Classifying Singularities).

- ☐ Isolated or not?
- ☐ If isolated, what order? (Lemma 4.5.11)

Remark (Function with Zeros Trick).
Let f be holomorphic with z_0, zero of order m. Then

$$f(z) = (z - z_0)^m g(z)$$

where $g(z) = \sum_{j=0}^\infty \frac{f^{(j+m)}(z_0)}{(j+m)!}(z - z_0)^j$, $g(z_0) \neq 0$.

Remark (Function with Poles Trick).
Let f be holomorphic except at z_0, pole of order k. Then

$$f(z) = (z - z_0)^{-k} H(z)$$

where $H(z) = \sum_{j=0}^\infty a_{j-k}(z - z_0)^j$, ***holomorphic***.

Definitions

Definition 1.2.2.
A ***neighbourhood*** of $z_0 \in \mathbb{C}$ is an open set containing z_0.

Lemma 1.3.8.
Let $f : \mathbb{C} \to \mathbb{C}$. The f is continuous $\Leftrightarrow$ the preimage $f^{-1}(U)$ is open for all open $U \subseteq \mathbb{C}$.

Definition 1.4.12.
Let $h : \mathbb{R} \to \mathbb{R}$. Then h is ***harmonic*** if it satisfies for all $(x, y) \in \mathbb{R}^2$ Laplace's equation:

$$\frac{\partial^2 h}{\partial x^2}(x, y) + \frac{\partial^2 h}{\partial y^2} = 0.$$

Definition 1.4.14.
Let $U \subseteq \mathbb{R}^2$ be open, $u : U \to \mathbb{R}$ harmonic. The function $v : U \to \mathbb{R}$ is the ***harmonic conjugate of*** u if $f = u + iv$ is holomorphic on U.

Definition 1.6.4.

$$\cos(z) \coloneqq \frac{\exp(iz) + \exp(-iz)}{2}, \quad \sin(z) \coloneqq \frac{\exp(iz) - \exp(-iz)}{2i}$$

Remark (Trigonometric Functions).

$$\tan \coloneqq \frac{\sin}{\cos}, \quad \cot \coloneqq \frac{1}{\tan}, \quad \sec \coloneqq \frac{1}{\cos}, \quad \csc \coloneqq \frac{1}{\sin}$$

Definition 1.6.9.

$$\cosh(z) \coloneqq \frac{\exp(z) + \exp(-z)}{2}, \quad \sinh(z) \coloneqq \frac{\exp(z) - \exp(-z)}{2}$$

Definition 1.7.6.
A ***branch cut*** L is a subset of complex plane removed s.t. multivalued function can be defined holomorphic on $\mathbb{C} \setminus L$. An endpoint of a branch cut is a ***branch point***.
The set $L_{z_0,\phi} = \{z \in \mathbb{C} : z = z_0 + re^{i\pi}, r \geqslant 0\}$ denotes a ***half-line***.
The set $D_{z_0,\phi} = \mathbb{C} \setminus L_{z_0,\phi}$ denotes the cut plane with branch point z_0 and along angle line at angle ϕ.

Definition 1.7.8.
Let $\phi \in \mathbb{R}$. We define the branch $\mathrm{Arg}_\phi(z)$ of the argument function s.t. $\phi < \mathrm{Arg}_\phi(z) \leqslant \phi + 2\pi$. This defines a branch of logarithm:
$\mathrm{Log}_\phi(z) = \ln|z| + i\,\mathrm{Arg}_\phi(z)$. N.B.: The ***principal branch*** is when $\phi = -\pi$.

Definition 1.8.1.
Let $\alpha, z \in \mathbb{C}$, $z \neq 0$. Then we define the ***α-power of*** z by $z^\alpha = \{\exp(\alpha w) : w \in \log(z)\}$.

Definition 1.8.4.
Let $q \in \mathbb{N}$, then the q values:

$$1^{1/q} = \{1, \omega, \omega^2, \ldots, \omega^{q-1}\}$$

where $\omega \coloneqq \exp(i2\pi/q)$, are the q roots of unity.

Definition 1.8.7.
The principal branch of logarithm defines the principal branch of z^α by $z^\alpha = \exp(\alpha\,\mathrm{Log}(z))$.

Definition 2.1.2.
Let $U \subseteq \mathbb{C}$ and $f : U \to \mathbb{C}$. f is ***conformal*** if f preserves angles.

Definition 2.2.1.
A ***Möbius transformation*** is a function of form

$$f(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$.

Definition 2.3.1.
The ***extended complex plane*** is the set $\tilde{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. where for $a \in \mathbb{C}$ and $b \in \mathbb{C} \setminus \{0\}$:

$$a + \infty = \infty, \quad b \cdot \infty = \infty, \quad \frac{b}{0} = \infty, \quad \frac{b}{\infty} = 0$$

For $f(z) = \frac{az+b}{cz+d}$ we define $f(-d/c) = \infty$ and $f(\infty) = a/c$.

Definition 2.4.1.

(i) ***Translation:*** $f(z) = z + b$, $b \in \mathbb{C}$ with matrix $\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)$;
(ii) ***Rotation:*** $f(z) = az$, $a = e^{i\theta} \in \mathbb{C}$ s.t. $|a| = 1$ with matrix $\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$;
(iii) ***Dilation:*** $f(z) = rz$, where $r > 0$ with matrix $\begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix}$;
(iv) ***Inversion:*** $f(z) = 1/z$ with matrix $\left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right)$.

A Möbius transformation ***fixes the point at infinity*** if $f(\infty) = \infty$. Only inversion does ***not*** fix the point at infinity.

Definition 2.5.5.
Let $z_1, z_2, z_3, z_4 \in \tilde{\mathbb{C}}$ be ***distinct*** points. The ***cross-ratio*** $[z_1, z_2, z_3, z_4]$ is the image z_1 under the Möbius transformation sending z_2, z_3, z_4 to $(1, 0, \infty)$.

Definition 3.2.7.
Let $\Gamma \subseteq \mathbb{C}$ be a regular curve. We define ***arclength*** $\ell(\Gamma)$ by:

$$\ell(\Gamma) \coloneqq \int_{t_0}^{t_1} |\gamma'(t)|\, dt = \int_{t_0}^{t_1} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

Definition 3.3.1.
Let $D \in \mathbb{C}$. We say D is a ***domain*** if D is ***open*** and any two points in D can be connected by a contour entirely in D.

Definition 3.4.1.
Let $\Gamma \subseteq \mathbb{C}$ be a contour. Then Γ is ***simple*** if it has no self-intersections.

Definition 3.4.6.
Let D be a domain. Then D is ***simply-connected*** if for all loops $\Gamma \in D$ we have $\mathrm{Int}(\Gamma) \subseteq D$.

Definition 4.3.1.
Let $z_0 \in \mathbb{C}$ and f holomorphic at z_0. The ***Taylor series*** of f ***centred at*** z_0 is:

$$\sum_{j=0}^\infty \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j$$

Definition .
Let $z_0 \in \mathbb{C}$. A ***Laurent series*** centred at z_0 is a series of the form:

$$\sum_{j=-\infty}^\infty a_j (z - z_0)^j$$

Definition 4.5.1.
Let $D \subseteq \mathbb{C}$ be a domain, $f : D \to \mathbb{C}$, $z_0 \in \mathbb{C}$. We say z_0 is a ***singularity*** if f is ***not*** holomorphic at z_0.
Singularity z_0 is ***isolated***, if $\exists R > 0$ s.t. f is holomorphic on $D'_R(z_0)$.

Definition 4.5.3.
Let $z_0 \in \mathbb{C}, U \subseteq \mathbb{C}$ be a neighbourhood of z_0, f holomorphic on U. Then z_0 is a ***zero*** of f if $f(z_0) = 0$.
Zero z_0 is ***zero of finite order*** if $\exists m \in \mathbb{N}$ s.t.

$$f(z_0) = f'(z_0) = \ldots = f^{(m-1)}(z_0) = 0$$

but $f^{(m)}(z_0) \neq 0$.
Singularity z_0 is ***isolated***, if $\exists R > 0$ s.t. $f(z) \neq 0$ for $z_0 \in D'_R(z_0)$.

Definition 4.5.7.
Let $z_0 \in \mathbb{C}$ be an isolated singularity of f, holomorphic on $D'_R(z_0)$ for some $R > 0$. Then $f(z) = \sum_{j=-\infty}^\infty a_j(z - z_0)^j$ on $A_{0,R}(z_0)$ (Laurent Series. If

(i) $\forall j < 0 : a_j = 0$, then z_0 is ***removable***;
(ii) $\forall j < -m : a_j = 0$ for some $m \in \mathbb{N}$ and $a_{-m} \neq 0$, then z_0 is ***pole of order m***;
(iii) $a_j \neq 0$ for infinitely many j, z_0 is ***essential***.

Definition 4.6.1.
Let $D \subseteq \tilde{D} \subseteq \mathbb{C}$ be domains, $f : D \to \mathbb{C}$ holomorphic. $F : \tilde{D} \to \mathbb{C}$ is an ***analytic continuation*** of f if $\forall z \in D : F(z) = f(z)$.

Definition 5.1.2.
Let $z_0 \in \mathbb{C}$ and f holomorphic on $D'_R(z_0)$ for some $R > 0$, with ***isolated*** singularity z_0. Then ***residue of*** f ***at*** z_0 is $\mathrm{Res}(f, z_0) = a_{-1}$, where a_{-1} is from Laurent Expansion of f.

Definition 5.2.1.
Let $D \subseteq \mathbb{C}$ be a domain. Function f is ***meromorphic*** on D if for all $z \in D$ either f has a pole of finite order or f is holomorphic at z.