DEs Formula Sheet

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Higher-Order ODEs

Exact form and Integrating factors

An ODE is in *exact form* if it's of the form $M(x,y) + N(x,y)\frac{dy}{dx} = 0$ with $M_y = N_x$, sometimes an ODE can be made exact with integrating factors to get $I\big[M(x,y)dx + N(x,y)dy\big] = 0$ for integrating factor I given by
$I = e^{-\int h(y)dy}$ if $\frac{M_y - N_x}{M} = h(y)$
$I = e^{-\int g(x)dx}$ if $\frac{N_x - M_y}{N} = g(x)$

If an ODE is exact then it defines a *conservative field*, whereby there is some kind of 'potential energy'.

The Wronskian

$$W[y_1, y_2, \ldots, y_n] := \begin{vmatrix} y_1 & \cdots & y_n \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix},$$

This determines whether or not the set of solutions $\{y_i\}$ forms a linearly independent basis; if $W[y_1, \ldots, y_n] = 0$ anywhere then it is zero everywhere, and the y_i are not independent, otherwise it's nowhere-zero and the y_i are linearly independent.

Systems of ODEs

Homogeneous Solution Method

If you have a homogeneous system of ODEs, ie of the from $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$, then you get $\mathbf{x}_h$ by finding the eigenvalues and vectors of the matrix A, and hence of the operator $\frac{d}{dt}$. Call the eigenvalues λ_1 and λ_2 with corresponding eigenvectors ξ_{λ_1} and ξ_{λ_2}, then we have the cases:

- $\lambda_1 \neq \lambda_2$ are real $\Rightarrow \mathbf{x}_h = c_1 e^{\lambda_1 t}\xi_{\lambda_1} + c_2 e^{\lambda_2 t}\xi_{\lambda_2}$.
- $\lambda_1^* = \lambda_2 = u + iv \Rightarrow \mathbf{x}_{h1} = e^{ut}\left(\mathbf{a}\cos(vt) + \mathbf{b}i\sin(vt)\right)$ then you split this into real and imaginary parts $\mathbf{x}_{h1} = \mathbf{h}_1 + i\mathbf{h}_2$ to get homogeneous solution $\mathbf{x}_h = c_1\mathbf{h}_1 + c_2\mathbf{h}_2$.
- $\lambda_1 = \lambda_2$ then you must find the generalised eigenvector η in the equation $(A - \mathbb{I}\lambda)\eta = \xi$ to get solution $\mathbf{x}_h = c_1 t e^{\lambda t}\xi + c_2 e^{\lambda t}\xi + c_3 e^{\lambda t}\eta$.

Undetermined Coefficients

If your inhomogeneous part is a polynomial, sine, cosine, or exponential

This is the same as for normal ODEs, except the coefficients are vectors, and the algebra is worse. For example, if your inhomogeneous term is $\mathbf{g}(t) = \begin{pmatrix} e^{at} \\ kt \end{pmatrix}$ then try $\mathbf{x}_{\text{par}} = \mathbf{a}e^{at} + \mathbf{b}t + \mathbf{c}$.

Variation of Parameters

For when you've ran out of other options.

If the coefficient matrix $P(t)$ in $\mathbf{x}' = P(t)\mathbf{x}$ isn't constant then seek solutions of the form $\mathbf{x}(t) = \Psi(t)\mathbf{u}(t)$ for fundamental matrix $\Psi(t)$, which is just the matrix of fundamental solutions in $\mathbf{x}_{\text{hom}}$. Remember that $\Psi' = P(t)\Psi$ by definition which will help cancel annoying terms along the way.

Diagonalisation

"Let $\mathbf{x} = T\mathbf{y}\ldots$" where T is the matrix of eigenvalues of A in $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$, then $T\mathbf{y}' = AT\mathbf{y} + \mathbf{g}(t)$ and so $\mathbf{y} = T^{-1}AT\mathbf{y} + T^{-1}\mathbf{g}(t)$, but $T^{-1}AT = D$ is diagonal (in fact the entries are the eigenvalues of A), so this gives us several very simple equations to solve one-by-one!

Critical Points and whatnot

If you have a two dimensional non-linear system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ with non-linear term $\mathbf{g}(t)$ then you can take the Taylor-expansion $F(x,y) \approx F(x_0,y_0) + (x - x_0)\partial_x F(x_0,y_0) + (y - y_0)\partial_y F(x_0,y_0)$ in the form of a Jacobian matrix (this is just a useful change of notation) to get:

$$J = \frac{\partial(F,G)}{\partial(x,y)} = \begin{pmatrix} \partial_x F & \partial_y F \\ \partial_x G & \partial_y G \end{pmatrix}.$$

The Jacobian in this case approximates the non-linear system in the new system $\mathbf{u}' = J\mathbf{u}$. If you're approximating, then you might like this lovely little table which tells you whether or not a critical point is stable or not (don't forget to *find eigenvalues* of the Jacobian)!

	Linear System		**Linear Approximation**	
Eigenvalues	**type of point**	**stability**	**type of point**	**stability**
$r_1 > r_2 > 0$	Node	Unstable	Node	Unstable
$r_1 < r_2 < 0$	Node	Asymptotically Stable	Node	Asymptotically Stable
$r_1 < 0 < r_2$	Saddle Point	Unstable	Node	Unstable
$r_1 = r_2 > 0$	Node (proper/improper)	Unstable	Node or Spiral	Unstable
$r_1 = r_2 < 0$	Node (proper/improper)	Asymptotically Stable	Node or Spiral	Stable
$r = \lambda \pm \mu$				
$\lambda < 0$	Spiral	Asymptotically Stable	Spiral	Asymptotically Stable
$\lambda > 0$	Spiral	Unstable	Spiral	Unstable
$\lambda = 0$	Center	Stable	Center/Spiral	*Undetermined*

Assessing non-linear stability

"Anything but Poincare-Bendixson!"

Try Lyapunov's method if they ask: V should be positive-definite, then if V' is negative (resp. positive) definite in some region containing the critical point then the system is asymptotically stable (resp. unstable) at that point. Negative semi-definite only guarantees stability, not asymptotic stability.

Maybe you can write the system in exact form, that is, using $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}$ by the chain rule. If you can solve this to get $H(x,y) = c$ then try and ignore higher-order terms (since they're small perturbations and hence fast-vanishing close to critical points) and translating the problem $(x,y) \mapsto (x - x_0, y - y_0)$. If the trajectory is closed or inwards-spiraling then the critical point is stable.

Lap-o-lace Transforms; Fellowship of the DEs

For DEs with non-constant coefficients.

What is a Laplace transform? You can think of it as a kind of continuous analogue to a power series: $\sum_{n=0}^{\infty} c(n)x^n \to \int_0^\infty f(n)x^t dt \to \int_0^\infty f(n)e^{-st}$ where the "$\to$" means "dodgy maths". Nonetheless it's very useful to think of it like this as it means we can see the transform as a kind of weighting to our function f, and we can then change from horrible differential-manifold land to nice algebra-land – turning differential equations into partial fractions! Here's a list of transforms:

$f(t)$	$F(t) := \mathcal{L}\{f\}(s)$	$f(t)$	$F(t) := \mathcal{L}\{f\}(s)$
$f(t)$	$\int_0^\infty e^{-st}f(t)dt$	$e^{at}\sin(bt)$	$\frac{b}{(s^2-a^2)+b^2}$
1	$\frac{1}{s}$	$e^{at}\cos(bt)$	$\frac{s-a}{(s^2-a^2)+b^2}$
e^{at}	$\frac{1}{s-a}$	$e^{at}t^n$	$\frac{n!}{(s-a)^{n+1}}$
t^n	$\frac{n!}{s^{n+1}}$	$u_c(t)$	$\frac{e^{-cs}}{s}$
$\sin(at)$	$\frac{a}{s^2+a^2}$	$\delta(t-c)$	$\lim_{\alpha\to c}\left(e^{-\alpha s}\right)$
$\cos(at)$	$\frac{s}{s^2+a^2}$	$f^{(n)}(t)$	$s^nF(s) - \sum_{k=1}^{n-1} s^{n-k}f^{(k)}(0)$
$\sinh(at)$	$\frac{a}{s^2-a^2}$	$(-t)^n f(t)$	$F^{(n)}(s)$
$\cosh(at)$	$\frac{s}{s^2-a^2}$		

Type of Shift	$f(t)$	$F(t) := \mathcal{L}\{f\}(s)$
s-shift	$e^{-cs}f(t)$	$F(s+c)$
t-shift	$u_c(t)f(t-c)$	$e^{-cs}F(s)$
s-derivative	$tf(t)$	$-F'(s)$
scaling	$f(ct)$	$\frac{1}{c}F(\frac{s}{c})$

Which brings me to my next point, the convolution integral. It's a rather strange and laborious truth that $\mathcal{L}(fg) \neq \mathcal{L}(f)\mathcal{L}(g)$, which is why we need to use something called the convolution, given by:

$$\textbf{CONVOLUTION INTEGRAL: } f \star g = \int_0^t f(\tau)g(\tau - t)d\tau.$$

which gives that $\mathcal{L}(f \star g) = \mathcal{L}(f)\mathcal{L}(g)$. Don't stress though, the Laplace transform is still **commutative**, **associative** and **distributive**.

Partial Differential Equations

Any differential equation that is not partial is impartial, and so doesn't really mind.

There is a 'one-size-fits-all' method to follow with PDEs:

1. "Let $u(x,t) = X(x)T(t)$ .."
2. Separate the variables, introducing a separation constant $-\lambda$.
3. Solve the corresponding ODE in $X(x)$, this gives a quantisation of λ and tells us the 'geometry' of the ODE
4. If there are infinitely many possible λ then label them $\{\lambda_1, \lambda_2, \dots\}$, and find corresponding eigen-functions $X_n(x)$.
5. Use this λ and solve the ODE in $T(t)$.
6. Recall $u = XT$ and write the general solution
7. If the system is inhomogeneous with infinitely many λ then you'll probably have to use Fourier series to solve it; letting $u_n = X_nT_n$ and thus $u = \sum_n X_nT_n$, find the Fourier coeffiecients of $L[u(x,t)] = g(x,t)$ where $g(x,t)$ is the inhomogeneous term.
8. Use initial/boundary conditions to find any coefficient you can!

There are three main types of PDE that you might come across in your day-to-day life, these are:

- **Heat Equation:** $u_t = \alpha^2 u_{xx}$.
- **Wave Equation:** $u_{tt} = a^2 u_{xx}$ or $u_{tt} = a^2(u_{xx} + u_{yy})$.
- **Laplace Equation:** $u_{xx} + u_{yy} = 0$ or $u_{rr} + \dfrac{u_r}{r} + \dfrac{u_{\theta\theta}}{r^2} = 0$.

If you are ever in doubt with PDEs then first step; relax. Think about what variables/coefficients etc you have and which ones you need to find then try and see if you can use orthogonality and the inner product in any useful way. Try and simplify the problem to what you'd do with linear algebra - differentials are just linear operators after all, there are often good analogues!

Fourier's Theorem

REMEMBER: *Check to see if f is an odd or even function - it'll speed things up!*

We deal only with *periodic functions* f so that for some $T \in \mathbb{R}$ we have $f(t+T) = f(t)$, this lets us worry only about a small interval $[-L, L]$, which we can then approximate f on using sine and cosine functions. First of all, we use an orthogonal basis $\{\cos(t), \sin(t), 1\}$ for some space with an inner product given by

$$\big(u(t), v(t)\big) = \int_{-L}^{L} u(t)v(t)dt,$$

where $T = 2L$. This is a continuous analogue of the dot product: $\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i$. Since we have the orthogonal basis $\{\cos(t), \sin(t), 1\}$ we have that

$$(\sin(t), \cos(t)) = (\cos(t), 1) = (\sin(t), 1) = 0,$$
$$\left(\cos\left(\frac{m\pi t}{L}\right), \cos\left(\frac{n\pi t}{L}\right)\right) = \left(\sin\left(\frac{m\pi t}{L}\right), \sin\left(\frac{n\pi t}{L}\right)\right) = (1,1) = \delta_{mn}.$$

Fourier's theorem is that any periodic function $f(t)$ can be written in the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi t}{L}\right) + b_n \sin\left(\frac{n\pi t}{L}\right)\right),$$

where we can find the coefficients using the orthogonality of the basis functions above:

$$\frac{a_0}{2} = (1, f) = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$
$$a_n = \left(\cos\left(\frac{n\pi t}{L}\right), f\right) = \frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi t}{L}\right)f(t)dt$$
$$b_n = \left(\sin\left(\frac{n\pi t}{L}\right), f\right) = \frac{1}{L}\int_{-L}^{L}\sin\left(\frac{n\pi t}{L}\right)f(t)dt$$

Parseval's theorem:

$$||f(t)||^2 = (f,f) = \int_{-L}^{L}|f(t)|^2 dt = L\left[\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2)\right],$$

which is an infinite-dimensional analogue to Pythagoras' theorem.

Nonhomogenous Boundary Conditions

If you have **non-homogeneous** boundary values then look for time-independent solutions $v(x)$ then re-write the PDE with the following map

$$u(x,t) = v(x) + \omega(x,t)$$

where if our original boundary conditions were $u(0,t) = A, u(L,t) = B$ then our new ones are $\omega(0,t) = 0$ and $\omega(L,t) = 0$. To find $v(x)$ just plug it into the PDE noting that $\partial_t v(x) = 0$ and that $v(0) = A$ and $v(L) = B$, and solve the ODE in $v(x)$. Then just solve the PDE in ω and apply the map above at the end!

For example, you may let $v(x) = \lim_{t\to\infty} u(x,t)$ be the final steady-state so that $v(x) = A + \frac{(B-A)}{L}x$ (like how a vibrating string must eventually come to rest). Thus $u(x,0) = u_{\text{init}}(x,0) - v(x)$ where $u_{\text{init}}(x,0)$ is the initial condition, this step removes any contribution made by this final steady-state. Then $u(x,t) = v(x) + \omega(x,t)$ where $\omega(x,t)$ is the solution to the corresponding homogeneous problem by the principle of superposition.

Conceptually what you're doing here is 'tilting' the underlying geometry. Remember that the boundary conditions determine the shape of the domain of the PDE, and so shifting these by A at one end and B at the other is just like tilting the PDE.

Sturm Liouville

Just like Fourier's theorem but more finicky and generally not as pleasant.

An S-L problem on $x \in [0,1]$ is one of the form

$$[p(x)y_x(x)]_x + q(x)y(x) = \lambda r(x)y(x), \quad p > 0, p', q \text{ and } r > 0 \text{ are cts},$$

the eigenvalues λ are what we're solving for, this is otherwise a very regular type of PDE, so you'd just have to solve it as you would any other. There are some shortcuts though; all eigenvalues of a Sturm-Liouville problem are real with orthogonal eigen-functions with respect to the inner product:

$$\langle u, v\rangle = \int_0^1 r(x)u(x)v(x)dx.$$

Regular homogeneous S-L problems have boundary conditions of the form $\alpha_1 y(0) + \alpha_2 y'(0) = 0$ and $\beta_1 y(1) + \beta_2 y'(1) = 0$, if these are not homogeneous then use the same method as for PDEs with nonhomogenous boundary conditions.

If the S-L problem *itself* isn't homogeneous, i.e. is of the form $[p(x)y_x(x)]_x + q(x)y(x) = \lambda r(x)y(x) + F(x)$, then solve the homogeneous equation then try and expand $\frac{F(x)}{r(x)}$ in terms of the eigenfunctions since it'll cancel with the $r(x)$ in the inner-product, and you can multiply through at the end (this is exactly why $r(x)$ is considered a weighting function by the way).

Really, the entire point of S-L theory is to generalise Fourier's methods which can be seen by noting that the basis $\left\{1, \sin\left(\frac{n\pi t}{L}\right), \cos\left(\frac{n\pi t}{L}\right)\right\}$ is just a set of solutions to the equation $y'' = \left(\frac{n\pi t}{L}\right)^2 y$ with $y(t+T) = y(t)$. Thus if you can solve a partial differential equation using Fourier series you just do *exactly the same* with S-L.

Bessel Stuffs

Bessel's equation is $y'' + \frac{1}{x}y' + \left(1 - \frac{m^2}{x^2}\right)y = 0$, and it's an example of a Sturm-Liouville problem whose eigenfunctions that *are not* sine and cosine functions. It has the two solutions

$$J_m(x) = \left(\frac{x}{2}\right)^m \sum_{k=1}^{\infty} (-1)^k \frac{x^{2k}}{k!(m+k)!}$$
$$Y_m(x) = \frac{2}{\pi}\log(x/2)J_m(x) + \sum_{k=0}^{\infty} \ldots x^{2k},$$

which can be found by taking the Laplace transform (using the s-derivative transform) and solving the resulting ODE in $Y(s) := \mathcal{L}\{y(x)\}(s)$.

Miscellaneous Theorems:

If you need a theorem then I've (hopefully) got your back

Superposition (*Worst superhero ever*): the full solution $\mathbf{x}$ of a system of ODEs is the combination of the homogeneous solution $\mathbf{x}_h$ and particular solution $\mathbf{x}_p$, this property is a consequence of the linearity of the differential operator $\frac{d}{dt}$.

d'Alembert's Method: If you have a solution to an ODE given by $f(t)$, then look for second solution of form $u(t)f(t)$ for some $u(t)$ to be determined. Plug this back into the original ODE and solve! That is: $y_{\text{par}} = \sum_j y_j(x) \int_{x_0}^x g(s)\frac{W_j[s]}{W[y_1(s),\ldots,y_n(s)]}ds$ where $g(s)$ is the inhomogeneous term, the y_j are the linearly independent homogeneous solutions, and $W_j[s]$ is the Wronskian with the j^{th} column removed.

Liouville's Theorem: $\dot{W} = \text{tr}(P(t))W \Rightarrow W = e^{\int_{t_0}^t \text{tr}(P(s))ds}W(t_0)$. And so if the Wronskian W is anywhere zero then it is everywhere-zero.

Exponential: $e^{At} = \sum_{n=0}^{\infty}\frac{(At)^n}{n!}$. If you have an initial condition $\mathbf{x}_0 := \mathbf{x}(t_0)$ then this can be used to get a fundamental matrix: $e^{At} = \Psi(t)\Psi^{-1}(t_0)\mathbf{x}_0$.

Exponential II: $det(e^{At}) = e^{t\cdot\text{tr}(A)}$, so the trace of A (sum of the diagonal) measures the rate of change of area (by differentiating both sides).

Stability: A critical point $\mathbf{x}_0$ is stable if $\forall\varepsilon > 0 : \exists\delta > 0$ such that for every solution $\bar{\phi}(t)$ we have $||\bar{\phi}(0) - \mathbf{x}_0|| < \delta \Rightarrow ||\bar{\phi}(t) - \mathbf{x}_0|| < \varepsilon$. It is called asymptotically stable if $\lim_{t\to\infty}\bar{\phi}(t) = \mathbf{x}_0$.

Locally Linear: A system $\mathbf{x}' = A\mathbf{x} + \mathbf{g}(t)$ is locally linear if $\mathbf{x} = \mathbf{0}$ is an isolated critical point and $\frac{||\mathbf{g}(t)||}{||\mathbf{x}||} \to 0$ as $\mathbf{x} \to \mathbf{0}$.

Liapunov Function: $V(x,y)$ is a lyapunnov function if it's positive definite in some region containing a critical point. If the derivative of V is negative-definite then the point is asymptotically stable, negative semi-definite only guarentees stability.

Basin of Attraction: If there is some region D_K for which there is a Lyapunov function $V(x,y) < K \in \mathbb{R}$ which is negative-definite then every solution which starts in D_K approaches the origin as $t \to \infty$. That is, the basin of attraction is $\text{dom}(V(x,y))$.

Closed trajectories contain critical points: If $\mathbf{x}$ is a solution to some system of ODEs which is closed then it encircles a critical point (THM 9.7.1).

Velocities and CPs If F_x and G_y have the same sign in a simply-connected domain D then there is no closed trajectory in D.

Poincare-Bendixson: (Used to assess stability; limit cycles) If $\bar{D}$ is the closure of some region D which contains no critical points but there is a $t_0 \in \mathbb{R}$ such that a solution to a system of ODEs is contained in $\bar{D}$ then that solution is either periodic, or approaching a periodic solution.

Fourier Convergence Theorem: If we use Fourier series to approximate $f(x)$ then the series converges to f wherever it is continuous and to $\frac{f(x^+)+f(x^-)}{2}$ wherever it isn't.

Lagrange's Identity:

$$\big\langle L[u], v\big\rangle - \big\langle u, L[v]\big\rangle := \int_0^1 \{L[u]v - uL[v]\} = -p(x)\big[u'v - v'u\big]_0^1.$$

For Sturm-Liouville problems the RHS is zero, which we can use to check whether or not a differential equation is *self-adjoint.* If the function-space defined by the boundary values and the differential equation leads to L being self-adjoint then we have (by Sturm-Liouville's theory) that all eigenvalues are real, and eigenfunctions are orthogonal!

Interestingly enough, Lagrange's identity is used to relate determinant to area, the LHS $=< L[u], v > - < u, L[v] >$ gives an equation for the 'areas' of two 'parallelepipeds,' one with sides u and $L[v]$, the other with sides v and $L[u]$. Thus the LHS tells you how L affects the area of the 'vectors' it's acting on, and the RHS is the determinant - I mean, just look at it!

Cylindrical coordinates: $(x,y,z) = (\rho\cos(\theta), \rho\sin(\varphi), z)$

Polar Conversion Tip: $r^2 = x^2 + y^2 \Rightarrow r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt}$

Model Answers

S-L with continuity and finiteness:
Given the complex function φ which satisfies $\varphi''(x) + (\lambda - V(x))\varphi(x) = 0$ with $V(x) = 0$ for $x \in (-\infty, -L)$ and $V(x) = -|V_0|$ for $x \in (-L, 0)$ along with $\varphi(0) = 0$ and $\varphi(x), \varphi'(x)$ are continuous at $x = -L$ then we solve for λ as follows. Show it's an S-L problem to assert $\lambda \in \mathbb{R}$, find when the inner product $(\varphi, \varphi) = \int_{-\infty}^{0} \varphi(x)\varphi^*(x)dx$ is finite (and hence normalisable) by breaking the ODE down into piecewise conditions of $V(x)$ and solving for $\varphi(x)$, eliminating any terms that make $\varphi(x) \to \pm\infty$ for $x \in (-\infty, 0)$. Finally solve the continuity condition at $x = -L$ by equating the functions $\varphi_I(-L) = \varphi_{II}(-L)$ (that is, the solutions to the ODEs in the different 'pieces' of V) and $\varphi_I'(-L) = \varphi_{II}'(-L)$ then asserting that the determinant of coefficients vanishes.

Bessel Bad Boys:
Say you use separation of variables and you end up seeing a horrible equation for the radius, something like

$$rR''(r) + 2R'(r) + \lambda r R(r) = 0,$$

then look for a sneaky substitution and solve it that way - in this case using $v(r) = rR(r)$ since the original PDE is just $\frac{1}{r}\frac{\partial^2 (ru)}{dr^2} = \frac{\partial^2 u}{\partial t^2}$. Failing this you'll just have to try and force it into the form of Bessel's equation and use the Bessel solution, using that $j_0(x) = \frac{\sin(x)}{x}$.

Assorted December 2017 Qs:
Question 2(b): "Show that $\mathcal{L}\{f(t) \star g(t)\}(s) = F(s)G(s)$": Swapping the order of integration yields $\mathcal{L}\{f(t) \star g(t)\}(s) = \int_0^\infty e^{-st}dt = \int_0^t f(t-\tau)g(\tau)d\tau = \int_0^\infty g(\tau)d\tau \int_\tau^\infty e^{-st} f(t-\tau)dt = \int_0^\infty e^{-s\tau} g(\tau)d\tau \int_0^\infty e^{-s(t-\tau)} f(t-\tau)d(t-\tau) = G(s)F(s)$.

Question 5 (b),(c):

$$L[y] = x^2y'' + 2xy' + (\lambda + 1/4)y = 0 \quad \text{with} \quad y(1) = y(e) = 0$$

Part B: "Find eigenfunctions": Try x^α giving the polynomial $\alpha(\alpha + 1) + \lambda + 1/4 = 0$ and so $\alpha = \frac{-1}{2} \pm i\sqrt{\lambda}$, thus our general solution is

$$\begin{aligned} y(x) &= x^{-1/2}\left(Ax^{i\sqrt{\lambda}} + Bx^{-i\sqrt{\lambda}}\right) \\ &= x^{-1/2}\left(C\cos\left(\sqrt{\lambda}\log(x)\right) + D\sin\left(\sqrt{\lambda}\log(x)\right)\right), \end{aligned}$$

since $x^\alpha = e^{\alpha\log(x)}$. Then do as you usually would (find λ and then solve).

Part C: "Show they're orthogonal for an inner product that you will define": Write out the ODE for $y_m(x)$ and $y_n(x)$ then cross-multiply, integrate and subtract to get $< y_m, L[y_n] > - < y_n, L[y_m] >$ (i.e. use Lagrange's identity) to, hopefully, get zero $\Rightarrow$ orthogonality!

Dec 2015 Q4:
"Find the general periodic bounded solution $u(r,\theta)$ to the ODE below":

$$u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0$$

for $\theta \in [-\pi, \pi]$ and $r \in (0, \rho)$, with $\rho > 0$ and $u(\rho, \theta) = f(\theta)$.

You use separation of variables by letting $u(r,\theta) = R(r)\Theta(\theta)$ and solving the ODEs in R ans Θ using *periodic boundary conditions* $\Theta(0) = \Theta(\pi)$ and $\Theta'(0) = \Theta'(\pi)$, solve for Θ first as to quantise the separation constant λ as $\{\lambda_n\}$.

Trig and Stuff

$$\begin{aligned}
\sin(x \pm y) &= \sin(x)\cos(y) \pm \sin(y)\cos(x) \\
\cos(x \pm y) &= \sin(x)\sin(y) \mp \cos(x)\cos(y) \\
\sin^2(x) &= \frac{1 - \cos(2x)}{2} \\
\cos^2(x) &= \frac{1 + \cos(2x)}{2} \\
\sinh(x) &= \frac{e^x - e^{-x}}{2} \\
\operatorname{csch}(x) &= \frac{1}{\sinh(x)} = \frac{2}{e^x - e^{-x}} \\
\cosh(x) &= \frac{e^x + e^{-x}}{2} \\
\operatorname{sech}(x) &= \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}} \\
\tanh(x) &= \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \\
\coth(x) &= \frac{1}{\tanh(x)} = \frac{e^x + e^{-x}}{ex - e{-x}} \\
\cosh^2(x) - \sinh^2(x) &= 1 \\
\tanh^2(x) + \operatorname{sech}^2(x) &= 1 \\
\coth^2(x) - \operatorname{csch}^2(x) &= 1
\end{aligned}$$

Integrals

$$\begin{aligned}
\int \sin^2(kx)dx &= \frac{x}{2} - \frac{\sin(2kx)}{4k} + \text{constant} \\
\int \cos^2(kx)dx &= \frac{x}{2} + \frac{\sin(2kx)}{4k} + \text{constant} \\
\int e^{ax}\sin(bx) &= \frac{a\sin(bx) - b\cos(bx)}{a^2 + b^2}e^{ax} \\
\int e^{ax}\cos(bx) &= \frac{b\sin(bx) + a\cos(bx)}{a^2 + b^2}e^{ax} \\
\int x^n e^{kx}dx &= \frac{x^n e^{kx}}{k} - \frac{n}{k}\int x^{n-1}e^{kx}dx
\end{aligned}$$

To normalise the function $u(x)$, set $\langle u(x), u(x) \rangle = 1$, which in Fourier series equates to $\int_{-L}^{L} u(x)u(x)dx = 1$.