Differential Equations
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DE Basics

Linear Nth Order ODE:

$$\frac{d^n y}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y(x) = g(x)$$

if $g(x) = 0$ then it is homogeneous
if $g(x) \neq 0$ then it is non-homogeneous

$$y_{gen}(x) = y_{hom}(x) + y_{par}(x)$$

If the Wronskian is zero then the solutions aren't linearly independent

$$W(y_1, ..., y_n) \equiv \begin{vmatrix} y_1 & y_2 & ... & y_n \\ y_1' & y_2' & ... & y_n' \\ ... & ... & ... & ... \\ y_1^{n-1} & y_2^{n-1} & ... & y_n^{n-1} \end{vmatrix}$$

if $k = \lambda + i\mu$ then $y_{hom} = e^{\lambda x}(c_1 \cos \mu x + c_2 \sin \mu x)$
The Chain Rule:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

Exact ODEs

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \iff \psi_x = \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \psi_y$$

Methods

D'Alemberts Theorem: find missing solution: Try
$y(x) = oldsolution \times u(x)$
Variation of Parameters (for non-homogeneous ODE's):
Looking for solution the form $y_{par} = \Sigma_{j=1}^n u_j(x) y_j(x)$
For the case where n = 3 we have
$\Sigma_j^n u_j' y_j = 0$, $\Sigma_j^n u_j' y_j' = 0$, $\Sigma_j^n u_j' y_j'' = g(x)$
Solve this system then integrate each u_j'
Then multiply by y_j to get $y_{par} = \Sigma_{j=1}^n u_j(x) y_j(x)$
aka:

$$y_{par} = \sum_j y_j(x) \int_{x_0}^x g(s) \frac{W_j[s]}{W[y_1(s), ... y_n(s)]} ds$$

Diagonalisation
Introducing change of variables $x = Ty$, so we have

$$Ty' = ATy + g(t) \implies \frac{dy}{dt} = Dy + T^{-1}g = Dy + h$$

This leads to a system of n decoupled equations which we solve by direct integration:

$$y_i' = r_i y_i + h_i \implies y_i(t) = c_i e^{r_i t} + e^{r_i t}\int_{t_0}^t e^{-r_i s} h_i(s) ds$$

General solution in the original variables equals $x(t) = Ty(t)$. In case the matrix is not diagonalisable, find its Jordan form and proceed in a similar way (with J instead of D). Note that we will need to integrate from the bottom up.

$$P^{-1}AP = D$$

$$J = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

Laplace Transform

map $y(x) \to F(s)$ through

$$F(s) = \int_\alpha^\beta K(s,x) y(x) dx$$

The Laplace Transform of $f(x)$ is defined for $x \in [0, \infty)$

$$F(s) = \mathcal{L}\{f\}(s) \equiv \int_0^\infty e^{-sx} f(x) dx = \lim_{T\to\infty} \int_0^T e^{-sx} f(x) dx$$

The Laplace Transform is a linear operation:

$$\mathcal{L}\{c_1 f_1(x) + c_2 f_2(x)\}(s) = c_1 \mathcal{L}\{f_1(x)\} + c_2 \mathcal{L}\{f_2(x)\}$$

If $f, f', ... f^{n-1}$ is continuous on $[0, \infty)$ and $\in E$ then:

$$\mathcal{L}\{f^n(x)\} = s^n \mathcal{L}\{f(x)\} - s^{n-1} f(0) - ... s f^{n-2}(0) - f^{n-1}(0)$$

$f(x)$	$F(s) = \mathcal{L}(f)(s)$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{(s-a)}$
$sin(ax)$	$\frac{a}{(s^2+a^2)}$
$cos(ax)$	$\frac{s}{(s^2+a^2)}$
$sinh(at)$	$\frac{a}{s^2-a^2}$
$cosh(at)$	$\frac{s}{s^2-a^2}$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$
$tcos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$tsin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$

Let $\mathcal{L}\{f(x)\}(s) = F(s)$ For $f \in E$:
s-shift: $\mathcal{L}\{e^{-cx} f(x)\}(s) = F(s+c)$
x-shift: $\mathcal{L}\{f(x-c)\}(s) = e^{-sc}F(s)$ if $c \leq 0$ and $f(x) = 0$ for $x < 0$
s-derivative: $\mathcal{L}\{xf(x)\}(s) = -F'(s)$
scaling: $\mathcal{L}\{f(cx)\}(s) = \frac{1}{c}F(\frac{s}{c})$, $F(sc) = \frac{1}{c}\mathcal{L}\{f(\frac{x}{c})\}$ if $c > 0$
Laplace transform of xe^x: $\mathcal{L}\{xe^x\}(s) = -(\frac{1}{s-1})' = \frac{1}{(s-1)^2}$
Change variable form t to $t' = t - c$: $\mathcal{L}\{u_c(t) f(t-c)\} = e^{-cs}F(s)$
The unit step function:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

Laplace transform of the unit step function ($c \geq 0$):

$$\mathcal{L}\{u_c(t)\}(s) = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st} dt = \frac{e^{-sc}}{s}, \ s > 0$$

For 'nice' functions $f(t)$.

$$\int_\infty^\infty \delta(t - t_0) f(t) dt = f(t_0)$$

$$\mathcal{L}\{\delta(t - t_0)\}(s) = e^{-st_0}$$

$$\mathcal{L}\{\delta(t)\}(s) = \lim_{\alpha\to 0} e^{-\alpha s}$$

$$\mathcal{L}^{-1}\{\lim_{\alpha\to 0} \frac{e^{-\alpha s}}{s^2 - a^2}\} = \frac{1}{a} u_0(t) sinh(at)$$

Laplace Proofs:
$\mathcal{L}\{y'\}(s) = \int_0^\infty e^{-st} y' dt = e^{-st} y(t)\big|_0^\infty + sY(s) = sY(s) - y(0)$
With $\mathcal{L}\{(-t)^n f(t)\}(s) = \frac{d^n F(s)}{ds^n}$, where $F(s) = \{f(t)\}(s)$ we get
$\mathcal{L}\{t^n e^{at}\}(s) = (-1)^n \frac{d^n}{ds^n}(s-a)^{-1} = \frac{n!}{(s-a)^{n+1}}$
Convolution:
$(f \star g)(t) \equiv \int_0^t f(t_1) g(t - t_1) dt_1$
$(\mathcal{L}\{f\})(\mathcal{L}\{g\}) = \mathcal{L}\{f \star g\}$
Example: $\mathcal{L}^{-1}\{\frac{\mathcal{G}(s)}{s^2 - a^2}\} = \frac{1}{a}(g(t) \star sinhat)$

Systems of DEs

First solution is $x^{(1)} = e^{\lambda t}\xi_\lambda$
Second solution is $x^{(2)} = te^{\lambda t}\xi_\lambda + e^{\lambda t}\eta$
Where $\xi_\lambda = (A - \lambda \mathbb{1})\eta$
Third solution is $x^{(3)} = \frac{t^2}{2}e^{\lambda t}\xi_\lambda + te^{\lambda t}\eta + e^{\lambda t}\zeta$
Where $\eta = (A - \lambda \mathbb{1})\zeta$

Critical Points

$$\text{Jacobian Matrix} = \begin{bmatrix} \partial_x F(x_0, y_0) & \partial_y F(x_0, y_0) \\ \partial_x G(x_0, y_0) & \partial_y G(x_0, y_0) \end{bmatrix}$$

Sometimes a nonlinear ODE has an exact phase portrait given by:

$$\frac{dy}{dx} = \frac{G(x,y)}{F(x,y)} \implies^{\text{(integrate)}} H(x,y) = c$$

Eigenvalues	**Critical** Points	**Stability**
$r_1 > r_2 > 0$	Node (source)	Unstable
$r_1 < r_2 < 0$	Node (sink)	Asymp. Stable
$r_2 < 0 < r_1$	Saddle	Unstable
$r_1 = r_2 > 0$	Proper/Improper Node	Unstable
$r_1 = r_2 < 0$	Proper/Improper Node	Asymp. Stable
$r_1, r_2 = \lambda \pm i\mu (\lambda > 0)$	Spiral (Focus)	Unstable
$r_1, r_2 = \lambda \pm i\mu (\lambda < 0)$	Spiral (Focus)	Asymp. Stable
$r_1 = i\mu, r_2 = -i\mu$	Centre	Stable

Almost Linear Systems the Proper/Improper bits become Node/Spiral Point. And the Centre becomes Centre or Spiral (Indeterminate)

Lyapunov Functions

Let $V(x,y)$ be defined on a domain D containing $(0,0)$
$V(x,y)$ is positive (negative) definite if $V(0,0) = 0$ and $E(x,y) > 0$ $\forall (x,y) \in D$ ($V(x,y) < 0$ $\forall (x,y) \in D$)
$V(x,y)$ is positive (negative) semi-definite if $V(0,0) = 0$ and $E(x,y) \geq 0$ $\forall (x,y) \in D$ ($V(x,y) \leq 0$ $\forall (x,y) \in D$)
Theorem Given an autonomous system with critical point $(0,0)$, if $\exists$ $V(x,y)$ continuous with continuous first partial derivatives, is positive definite then $(0,0)$ is:
Asymptotically Stable: If $\frac{dV}{dt}$ is negative definite on some domain D containing $(0,0)$
Stable (at non-linear level): If $\frac{dV}{dt}$ is negative semi-definite
Theorem Given an autonomous system with critical point $(0,0)$, assume $\exists$ $V(x,y)$ continuous with continuous first partial derivatives, such that $V(0,0) = 0$ and that in every neighbourhood of $(0,0)$ there exists at least one point (x_1, y_1) where $E(x_1, y_1)$ is positive (negative). If there exists some domain D containing $(0,0)$ where $\frac{dV}{dt}$ is positive definite (negative definite) on D, then (0,0) is an **unstable critical point**.

$$\frac{dV}{dt} = x'\partial_x V + y'\partial_y V$$

Limit Cycles

Limit cycles are periodic solutions s.t at least one other non-closed trajectory asymptotes to them as $t \to \infty$ (or $-\infty$ or both)
Theorem Let $F(x,y), G(x,y)$ have continuous first partial derivatives in some domain D. A closed trajectory must enclose at least one critical point. If it encloses only one, it cannot be a saddle point. (i.e no critical points in D implies no closed trajectories in D; if there exists a unique critical point in D and it is a saddle implies no closed trajectories in D).
Theorem Let $F(x,y), G(x,y)$ have continuous first partial derivatives in some simply connected domain D (i.e without holes). If $\partial_x F + \partial_y G$ has the same sign in D $\implies$ there are no closed trajectories in D.
Poincaré-Bendixon Theorem Let $F(x,y), G(x,y)$ have continuous first partial derivatives in some domain D. Let D_1 be a bounded subdomain of D and let R consist of D_1 and its boundary. Suppose R has no critical points. If $\exists$ a trajectory $(x(t), y(t))$ staying in R $\forall t \geq t_0 \implies$ either the solution is periodic (closed trajectory) or it spirals towards one. Either way there exists a closed trajectory.

$$rr' = xx' + yy'$$

Example: $\dot{r} = r^2(1-r^2), \dot{\theta} = 1$
$r = 1$, $\theta = t + t_0$ corresponds to a limit cycle. Notice that for $r > 1$, $\dot{r} < 0$, whereas for $r < 1$, $\dot{r} > 0$. Thus, the cycle $r = 1$ is stable. Can check this behaviours by plotting the trajectories not corresponding to periodic solutions.

$$-\frac{1}{r} + \frac{1}{2}log\frac{1+r}{1-r} = t + c$$

sldls

Sturm-Liouville Boundary Problems

S-L form is $L[y] + \lambda y = 0$ with $\mathcal{L}[y] \equiv -(p(x)y')' + q(x)y(x)$
Modified Inner Product: $\langle \phi_1, \phi_2 \rangle \equiv \int_0^1 r(x)\phi_1(x)\phi_2(x)dx$ if $\langle \phi_1, \phi_2 \rangle = 0$ then ϕ_1 and ϕ_2 are orthogonal with respect to the inner product.
Using normalized eigenfunctions Φ_n we can expand $f(x) = x = \Sigma_{n=1}^{\infty} a_n \Phi_n$
Over $0 \leq x < 1$ using orthonormality of inner product we find:
$a_n = \int_0^1 f(x)\Phi_n$

Fourier Analysis

Fourier Series of $f(x)$:

$$f(x) = \frac{a_0}{2} + \Sigma_{n=1}^{\infty}(a_n cos(\frac{n\pi x}{L}) + b_n sin(\frac{n\pi x}{L}))$$

Where $T = 2L$ or $f(x + 2L) = f(x)$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos(\frac{n\pi x}{L})dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin(\frac{n\pi x}{L})dx$$

Function is even if $f(-x) = f(x)$ and only have cosine coefficient series
Function is odd is $f(-x) = -f(x)$ and only have sine coefficient series

Partial Differential Equations

We have $a^2\partial_{xx}^2 u + \beta u = \partial_t u$
Now with non-homogeneous boundary conditions $u(0,t) = T_1$, $u(L,t) = T_2$, look for $v(x)$ to solve
$a^2 v''(x) + \beta v(x) = 0$, $v(0) = T_1$, $v(L) = T_2$
Then determine solution using $u(x,t) = v(x) + \omega(x,t)$

Identities

- $sinh(x) = \dfrac{e^x - e^{-x}}{2}$
- $cosh(x) = \dfrac{e^x + e^{-x}}{2}$
- $cosx = \dfrac{e^{ix} + e^{-ix}}{2}$
- $sinx = \dfrac{e^{ix} - e^{-ix}}{2i}$
- $cos\mu = 0 \implies \mu_n = (2n-1)\frac{\pi}{2}$
- 1-D Heat Eq: $\partial_t u = \alpha^2 \partial_x^2 u$
- $\int sin^2 axdx = \frac{x}{2} - \frac{sin2ax}{4a}$
- $\int cos^2 axdx = \frac{x}{2} + \frac{sin2ax}{4a}$