Differentiable Manifolds: Formula Sheet

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These are my notes for the "*Differentiable Manifolds*" course at Edinburgh University given by José Figueroa-O'Farrill in 2019. There will undoubtedly be many mistakes in this formula sheet, and it is far from finished: given the time and energy I would add proofs of theorems, answers to exercises and workshops, and various insights. If you notice any mistakes please email me at william.bevington@zoho.eu.

Contents

The Definition of a Maifold

Definition 1.1: Coordinate Chart

A **coordinate chart** on a set M is a pair (U, φ) where $U \subseteq M$ and $\varphi : U \to \varphi(U)$ is a bijection onto an open subset of $\mathbb{R}^n$.

Writing $\varphi(a) = (x^1(a), \dots, x^n(a))$ for $a \in U$ we get **local coordinate functions** x^i, giving a **local chart** $(U, x^1, \dots, x^n)$.

Definition 1.2: Smooth Function

Let $V \subseteq \mathbb{R}^n$. A function $f : V \to \mathbb{R}^m$ is **differentiable** at $a \in V$ if there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ so that

$$\lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{||h||} - \frac{L}{||h||} \right) = 0.$$

We say that L is the **derivative of f at a** and denote it $L = (df)_a$. Relative to standard bases in $\mathbb{R}^n$ and $\mathbb{R}^m$ we have a matrix representation of L via the **Jacobian Matrix**,

$$Df_a = \begin{pmatrix} \left.\frac{\partial f^1}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial f^1}{\partial x^n}\right|_a \\ \vdots & \ddots & \vdots \\ \left.\frac{\partial f^m}{\partial x^1}\right|_a & \cdots & \left.\frac{\partial f^m}{\partial x^n}\right|_a \end{pmatrix}$$

where $f(x^1, \dots, x^n) = \left(f^1(x^1, \dots, x^n), \dots, f^m(x^1, \dots, x^n)\right)$. If f is differentiable at each $a \in V$ then $Df : a \mapsto Df_a$ defines a continuous function. We say f is **smooth** if $D^n f$ exists for all $n \in \mathbb{N}$.

Theorem 1.3: Chain Rule

If g and f are smooth composable functions then

$$d(g \circ f)_a = (dg)_{f(a)} \circ (df)_a.$$

Definition 1.4: Atlas

Let Λ be some indexing set. An n-dimensional **coordinate atlas** on a set M is a collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ of coordinate charts such that

1. $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ so that $\{U_\alpha\}$ covers M,
2. $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open in $\mathbb{R}^n$ for all $\alpha, \beta \in \Lambda$, and
3. for all $\alpha, \beta \in \Lambda$, the **transition functions** $\varphi_\beta \circ \varphi_\alpha^{-1}(U_\alpha \cap U_\beta)$ is a smooth function between two open subsets of $\mathbb{R}^n$.

Definition 1.8: Compatible Atlases

Two atlases $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \Lambda}$ and $\{(V_\beta, \psi_\beta)\}_{\beta \in \Phi}$ are **compatible** if their union is an atlas.

Definition 1.10: Differentiable Structure

A **differentiable structure** on a set M is an equivalence class of compatible atlases.

Definition 1.11: Manifold

A **differentiable manifold** is a set M together with a differentiable structure.

Definition 1.17

Let M be a manifold with atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. A subset $V \subseteq X$ is **open** if for all $\alpha \in \Lambda$ then $\varphi_\alpha(V \cap U_\alpha)$ is open in $\mathbb{R}^n$.

Theorem 1.18: Manifold Topology

The open subsets of a manifold M define a topology on M

Definition 1.19

A manifold M is **second-countable** if it admits a countable atlas; i.e. an atlas with at most countably many charts.

Definition 1.25: Compact/Connected Manifold

Let M be a manifold with atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$. Then M is **compact** if *every* atlas has a finite sub-atlas, and M is **compact** if given any two $a, b \in M$ there is a finite set of charts $\{(U_i, \varphi_i)\}_{i=1,\dots,N}$ such that $a \in U_1$, $b \in U_N$ and $U_i \cap U_{i+1} \neq \emptyset$ for $i = 1, 2, \dots, N-1$.

Theorem 1.26: Inverse Function Theorem

Let $U \subseteq \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}^n$ be a smooth function with $(df)_a : \mathbb{R}^n \to \mathbb{R}^n$ invertible at $a \in U$. Then there exist neighbourhoods $a \in V$ and $f(a) \in W$ such that $f(V) = W$ and f has a smooth inverse on W.

Definition 1.27

Let $U \subseteq \mathbb{R}^{m+n}$ be open, $F : U \to \mathbb{R}^m$ be smooth, and $c \in \mathbb{R}^m$. We say that c is a **regular value** of F if for all $a \in F^{-1}(c)$ the derivative $(dF)_a : \mathbb{R}^{m+n} \to \mathbb{R}^m$ is surjective.

Theorem 1.27: Level Set

Let $U \subseteq \mathbb{R}^{m+n}$ be open and $F : U \to \mathbb{R}^m$ be smooth with $c \in \mathbb{R}^m$ as a regular value. Then $F^{-1}(c)$ is a smooth n-dimensional Hausdorff second-countable manifold.

Remark: *definition 1.29 gives that $F^{-1}(c)$ has dimension m.*

Definition 1.29: Regular Sub-Manifold

A subset $S \subseteq M$ of an n-dimensional manifold M is a **regular sub-manifold**[a] of M with dimension $k \leq n$ if for every $p \in S$ there is a chart (U, φ) of M with $p \in U$ and $U \cap S$ is defined by the vanishing of $n - k$ of the coordinate functions.

[a] Equally a regular sub-manifold is one with atlas $\{(U_\alpha \cap S, \varphi_\alpha)\}_{\alpha\in\Lambda}$.

Maps Between Manifolds

Definition 2.1: Smooth Map

A map $F : M \to N$ between an m-dimensional manifold M and an n-dimensional manifold N is **smooth** if for each $a \in M$ and chart (U, φ) in M and chart (V, ψ) in N with $a \in U$ and $F(a) \in V$ the composite

$$\psi \circ F \circ \varphi^{-1} : \varphi\Big(F^{-1}(V) \cap U\Big) \to \mathbb{R}^n$$

is smooth as a map from $\mathbb{R}^m$ to $\mathbb{R}^n$.

Definition 2.5: Diffeomorphism

A **diffeomorphism** $F : M \to N$ is a smooth bijection with smooth inverse.

Definition 2.8: Lie Group

A **Lie Group** is a smooth manifold G together with

- an element $e \in G$,
- a map $\mu : G \times G \to G$,
- a map $\iota : G \to G$,

such that

- `(Identity):` $\mu(e, a) = \mu(a, e) = a$ for all $a \in G$,
- `(Inverse):` $\mu(a, \iota(a)) = \mu(\iota(a), a) = e$ for all $a \in G$,
- `(Associativity):` $\mu(\mu(a, b), c) = \mu(a, \mu(b, c))$ for all $a, b, c \in G$.

We will often write $a^{-1} = \iota(a)$ and $ab = \mu(a, b)$.

Definition 2.10

Let G be a Lie group. For all $a, b \in G$ we have diffeomorphisms $\lambda_a : G \to G$ and $\rho_a : G \to G$ given by $\lambda_a(b) = \mu(a, b) = \rho_b(a)$.

Definition 2.11: Smooth Functions

A special case of smooth maps are **smooth functions**, which are smooth maps $f : M \to \mathbb{R}$. We denote the *commutative, associative* $\mathbb{R}$*-algebra* (hence an $\mathbb{R}$-module) of smooth functions on a manifold M as $C^\infty(M)$.

Definition 2.17: Cotangent Space

Let M be a smooth manifold. Denote by $Z_a(M) \subset C^\infty(M)$ the set of smooth functions whose derivative[a] vanish at $a \in M$. The **cotangent space** of M at a is the quotient vector-space

$$T_a^* M := C^\infty(M)/Z_a.$$

The **derivative** $(df)_a$ of $f \in C^\infty(M)$ at a is the image of f is the image under the canonical quotient map.

[a] We haven't actually defined what a derivative is yet, but $f \in Z_a$ if and only if the derivative $f \circ \varphi^{-1}$ vanishes at $\varphi(a)$, by the chain rule. Here $f \circ \varphi^{-1}$ is a real-valued function, so we have a definition of (vanishing) derivatives.

Theorem 2.20

Let M be an n-dimensional manifold, then:

- for all $a \in M$ the cotangent space T_a^*M is an n-dimensional real vector space,
- if (U, φ) is a coordinate chart around a with local coordinates $(x^1, \dots, x^n)$ then $(dx^1)_a, \dots, (dx^n)_a$ is a basis for T_a^*M, and
- if $f \in C^\infty(M)$ then
$$(df)_a = \sum_{i=1}^{n} \partial_i (f \circ \varphi^{-1})\Big|_{\varphi(a)} (dx^i)_a,$$
where ∂_i is the i-th derivative with respect to the i-th coordinate on $\mathbb{R}^n$.

Definition 2.22: Tangent Space

Let M be an n-dimensional manifold with $a \in M$. The **tangent space** T_aM to M at a is the dual vector space to T_a^*M, i.e. the linear functions $C^\infty(M) \to \mathbb{R}$ sending Z_a to zero.

Definition 2.23: Directional Derivative

A **directional derivative** at $a \in M$ is a linear map $X_a : C^\infty(M) \to \mathbb{R}$ satisfying the *Liebniz rule*:
$$X_a(fg) = f(a)X_a(g) + g(a)X_a(f).$$

Theorem 2.24

Let X_a be a directional derivative at $a \in M$ and $f \in Z_a$ then $X_a f = 0$. Hence $X_a \in T_aM$, and
$$\left(\frac{\partial}{\partial x^1}\Big|_a, \dots, \frac{\partial}{\partial x^n}\Big|_a \right),$$
is the **canonical dual basis** to $\Big((dx^1)_a, \dots, (dx^n)_a\Big)$.

Theorem 2.25

If c is a **smooth curve** in M passing through $a \in M$, i.e. a smooth map $c : (-\varepsilon, \varepsilon) \to M$ with $c(0) = a$. We define the **velocity** of c at a to be the function
$$c'(0) : C^\infty(M) \to \mathbb{R}, \qquad c'(0)f = \frac{d}{dt}(f \circ c)(t)\Big|_{t=0}.$$
Then $c'(0) \in T_aM$. In fact every $X_a \in T_aM$ is of the form $c'(0)$ for some curve c through a.

Definition 2.27: Push Forward/Derivative

The **derivative** (or '*push-forward*') at $a \in M$ of a smooth map $F : M \to N$ is the linear map

$$(F_*)_a : T_aM \to T_{F(a)}N, \qquad (F_*)_a(X_a)(f) = X_a(f \circ F).$$

If $X_a = c'(0)$ then $(F_*)_a(X_a)(f) = (F \circ c)'(0)(f)$, and

$$(F_*)_a \left(\frac{\partial}{\partial x^i}\Big|_a \right) = \sum_{j=1}^{n} \frac{\partial F}{\partial x^i}\Big|_a \frac{\partial}{\partial y^j}\Big|_{F(a)}.$$

Definition 2.28: Submersion, Immersion, Embedding

Let $F : M \to N$ be a smooth map between an m-dimensional manifold M and an n-dimensional manifold N, then

- F is a **submersion** if $(F_*)_a$ is surjective for all $a \in M$
- F is a **immersion** if $(F_*)_a$ is injective for all $a \in M$
- F is an **embedding** if F is a homeomorphism onto its image, and is an immersion.

Warning: injectivity/surjectivity of F says nothing about injectivity/surjectivity of $(F_*)_a$.

Definition 2.29: Embedded Sub-Manifold

A manifold M is an **embedded sub-manifold** of a manifold N if there is an embedding $\iota : M \to N$.

Definition 2.30: Regular Value

Let $F : M \to N$ be a smooth map between manifolds. We say $c \in N$ is a **regular value** of F if for all $a \in M$ with $F(a) = c$ the derivative $(F_*)_a : T_aM \to T_cN$ is surjective.

Theorem 2.31: Regular Sub-Manifold

Let $F : M^{m+n} \to N^m$ be a smooth map between manifolds and $c \in N$ be a regular value of F. Then $F^{-1}(c)$ is an n-dimensional embedded sub-manifold of M and for all $a \in F^{-1}(c)$

$$T_aF^{-1}(c) = \ker(F_*)_a.$$

The Tangent Bundle and Vector Fields

Theorem 3.1: Tangent Bundle

Let M be a manifold, we define the **tangent bundle** as

$$TM := \bigsqcup_{a\in M} T_aM,$$

with projection map $p : TM \to M$ sending $v \in T_aM$ to $a \in M$. Suppose that $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is an atlas for M, then we have seen that $\left(\frac{\partial}{\partial x^1}\Big|_a, \dots, \frac{\partial}{\partial x^n}\Big|_a,\right)$ is a basis for T_aM, so we have a bijection

$$\psi : U \times \mathbb{R}^n \to TU := \bigsqcup_{a\in U} T_aM, \qquad \psi(a, v^1, \dots, v^n) = \sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\Big|_a.$$

Then $\{(TU_\alpha, \Psi_\alpha)\}_{\alpha\in\Lambda}$ with

$$\Phi = (\varphi \times \mathrm{id}) \circ \psi^{-1} : TU \to \varphi(U) \times \mathbb{R}^n, \qquad \Psi\left(\sum_{i=1}^n v^i \frac{\partial}{\partial x^i}\right) = (x^1, \dots, x^n, v^1, \dots, v^n)$$

is an atlas for TM.

Theorem 3.2

The tangent bundle TM of a manifold M is Hausdorff and second-countable (provided M is).

Definition 3.4: Vector Field

A **vector field** on a manifold M is a smooth map $\mathfrak{X} : M \to TM$ such that $p \circ X = \mathrm{id}_M$, in other words $X(a) \in T_aM$. The set of all vector fields is denoted $\mathfrak{X}(M)$, and we use the abuse of notation $X \text{“}=\text{''} (\Psi \circ X \circ \varphi^{-1})(x^1, \dots, x^n) = (x^1, \dots, x^n, X^1(x), \dots, X^n(x))$.

Any vector field $X : M \to TM$ is an embedding so that $X(M)$ is an embedded sub-manifold of TM diffeomorphic to M (exercise 4.6).

Definition 3.5

The **zero section** of TM is the map $s : M \to TM$ given by the vector field $X = 0$. All sections of TM are vector fields[a].

[a]Though we will only define what a section is later

Theorem 3.7

Given a smooth map $F : M \to N$ from an m-dimensional manifold M to an n-dimensional manifold N the push-forwards $(F_*)_a : T_aM \to T_{F(a)}N$ assemble to a smooth map $F_* : TM \to TN$. In other words, there is a functor F_* from the category of manifolds and smooth maps to its self.

We can do a similar trick for the **cotangent bundle** T^*M, with a couple of differences. There is no functor $T^*F : T^*M \to T^*N$, but we can pull back sections of T^*N to sections of T^*M (see workshop 3).

Definition 3.8

Any $\mathbb{R}$-linear transformation X of $C^\infty(M)$ obeying the *Leibniz rule*

$$X(fg) = f\,X(g) + g\,X(f)$$

is called a **derivation** of $C^\infty(M)$.

Theorem 3.10

A transformation $X : C^\infty(M) \to C^\infty(M)$ is a vector field if and only if it is a derivation.

Definition 3.11

The **lie bracket** of two vector fields X and Y is defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)),$$

and is its self a vector field satisfying (exercise 4.13)

- $[-, -]$ is $\mathbb{R}$-bilinear,
- $[X, Y] = -[Y, X]$ ('*skew-symmetry*'),
- $[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$ ('*Jacobi identity*'), and
- $[X, fY] = f[X, Y] + X(f)Y$.

Definition 3.14

Let $F : M \to N$ be a smooth map between differentiable manifolds. We say that a vector field $X \in \mathfrak{X}(M)$ is F**-related** to $Y \in \mathfrak{X}(N)$ if for all $a \in M$ the identity $(F_*)_a(X_a) = Y_{F(a)}$ holds.

Theorem 3.15

A vector field $X \in \mathfrak{X}(M)$ is F-related to $Y \in \mathfrak{X}(M)$ if and only if $X(f \circ F) = (Yf) \circ F$ for all $f \in C^\infty(M)$.

Definition 3.18

A **one parameter group of diffeomorphisms** of a manifold M is a smooth map $\psi : \mathbb{R} \times M \to M$ with $\psi(t, a) = \psi_t(a)$ such that for all $s, t \in \mathbb{R}$:

- $\psi_t : M \to M$ is a diffeomorphism,
- $\psi_0 = \mathrm{id}_M$, and
- $\psi_{s+t} = \psi_s \circ \psi_t$.

Suppose that ψ_t is a one parameter group of diffeomorphisms of M and $f \in C^\infty(M)$ then for all $a \in M$ the function $\mathbb{R} \to \mathbb{R}$ given by $t \mapsto f(\psi_t(a))$ is smooth and hence ψ_t defines a vector field

$$X_a(f) = \frac{d}{dt} f(\psi_t(a))\Big|_{t=0}.$$

In local coordinates $\psi_t(x^1, \ldots, x^n) = (y^1(t, x), \ldots, y^n(t, x))$ we have

$$X_a(f) = \sum_{i=1}^{n} \left(\frac{\partial y^i}{\partial t}\Big|_{t=0} \right) \frac{\partial f}{\partial x^i}.$$

Definition 3.19

An **integral curve** of a vector field $X \in \mathfrak{X}(M)$ is a smooth map $\psi : (\alpha, \beta) \to M$ such that for all $t \in (\alpha, \beta)$

$$(\psi_*)_t \left(\frac{d}{dt} \right) = X_{\psi(t)}.$$

In fact, if (U, φ) is a chart of M with local coordinates $(x^1, \ldots, x^n)$ and $X = \sum_{i=1}^n X^i(x) \frac{\partial}{\partial x^i} \in \mathfrak{X}(U)$ then ψ is a smooth curve satisfying

$$\psi_* \left(\frac{d}{dt} \right) = \sum_{i=1}^{n} X^i(x(t)) \frac{\partial}{\partial x^i},$$

giving a first-order system of ODEs $\frac{dx^i}{dt} = X^i(x(t))$. Thus a vector field is an ODE on that manifold and solving that ODE is equivalent to finding its integral curves.

Theorem 3.21

Let $V \subset \mathbb{R}^n$ be open with $p_0 \in V$ and $f : V \to \mathbb{R}^n$ smooth. Then the initial value problem

$$\frac{dy}{dt} = f(y), \qquad y(0) = p_0$$

has a unique smooth solution $y : (\alpha, \beta) \to V$ where α, β depend on p_0 and (α, β) is the maximal interval containing 0 on which y is defined.

Theorem 3.22

Let $V \subseteq \mathbb{R}^n$ be open and $f : V \to \mathbb{R}^n$ be smooth. For each $p_0 \in V$ there exists $W \subset V$ open with $p_0 \in W$, $\varepsilon > 0$ and smooth

$$y : (-\varepsilon, \varepsilon) \times W \to V$$

such that

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q)), \qquad y(0, q) = q \qquad \forall (t, q) \in (-\varepsilon, \varepsilon) \times W.$$

Definition 3.23

Theorem 3.22 gives that if $X \in \mathfrak{X}(U)$ then for every $a \in U$ there is a $W \subset U$ open with $a \in W$, $\varepsilon > 0$ and smooth $\psi : (-\varepsilon, \varepsilon) \times W \to U$ such that for all $p \in W$ we have that $\psi_t(p) = \psi(t, p)$ is an integral curve. We call ψ the **local flow generated by** X. If ψ is defined on $\mathbb{R} \times M$ then we call it **global flow**.

Definition 3.24: Lie Derivative

If ψ_t is the local flow generated by $X \in \mathfrak{X}(M)$ then for all $f \in C^\infty(M)$ we saw previously that

$$X(f) = \frac{d}{dt}(f \circ \psi_t)\Big|_{t=0} \in C^\infty(M).$$

We call this the **Lie derivative** $\mathcal{L}_X$ along X. Equivalently:

$$(\mathcal{L}_X Y)(f) = [X, Y](f).$$

Vector Bundles

Definition 4.1: Tensor Product

Let V, W be two finite-dimensional real vector spaces. Their **tensor product** $V \otimes W$ is the vector space defined by the set of linear maps $\otimes : V \times W \to V \otimes W$ sending $(v, w) \mapsto v \otimes w$ satisfying the following universal property:

Given a bilinear map $B : V \times W \to U$ into a vector space U there is a *unique* linear map $\beta : V \otimes W \to U$ such that the following commutes:

$$\begin{array}{ccc} V \times W & \xrightarrow{B} & U \\ \downarrow \otimes & \nearrow_{\beta} & \\ V \otimes W & & \end{array}$$

It turns out that $V \otimes W$ is just the dual to the vector space $\mathrm{Bil}(V, W)$ of bilinear maps, and there is a natural isomorphism $\mathrm{Hom}(V \otimes W, U) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, U))$.

Definition 4.2

Denote by $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$. Then the **tensor algebra** of V has underlying space

$$T(V) = \bigotimes_{k=0}^{\infty} V^{\otimes k}$$

whose elements are finite sums $\lambda + v + \sum v_i \otimes v_j + \cdots + \sum v_{i_1} \otimes \cdots v_{i_p}$ and if $u, v \in T(V)$ then their product is just $u \otimes v$. This algebra is associative.

Vectors in $T^r_s(V) = V^{\otimes r} \otimes (V^*)^{\otimes s}$ are called (r, s)**-tensors**, we can understand (r, s)-tensors as linear maps $V^{\otimes s} \to V^{\otimes r}$ by the isomorphism $V \otimes V^* \cong \mathrm{End}(V)$. That is, $T^r_s(V) \cong \mathrm{Hom}(V^{\otimes s}, V^{\otimes r})$.

Definition 4.5: Vector Bundle

A real **vector bundle or rank** m consists of

1. a manifold M, called the **base space**,
2. a manifold E, called the **total space**,
3. a smooth surjection $\pi : E \to M$ called the **projection**

such that

4. for all $a \in M$ the **fibre** $\pi^{-1}(a)$ is isomorphic as a vector space to $\mathbb{R}^m$
5. for all $a \in M$ there is an open neighbourhood U of a and a diffeomorphism $\varphi_U : \pi^{-1}(U) \to U \times \mathbb{R}^m$ (called the **local trivialisation**) such that φ_U maps the vector space $\pi^{-1}(a)$ isomorphically to the vector space $\{a\} \times \mathbb{R}^m$, and
6. If (U, φ_U) and (V, φ_V) are two local trivialisations with $U \cap V \neq \emptyset$ then

   $$\varphi_U \circ \varphi_V^{-1} : (U \cap V) \times \mathbb{R}^m \to (U \cap V) \times \mathbb{R}^m,$$

   takes the form $(a, v) \mapsto (a, g_{UV}(a)v)$ where the **transition function** $g_{UV} : U \cap V \to \mathrm{GL}(m, \mathbb{R})$ is smooth.

A vector bundle is **trivial** if the neighbourhood in (4) can be taken to be all of M. If the rank of the vector bundle is one, we call it a **line bundle**.

Definition 4.6

Let $\pi : E \to M$ be a real rank m vector bundle, then $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ is a **trivialising cover** if $M = \bigcup_{\alpha\in\Lambda} U_\alpha$ and $(U_\alpha, \varphi_\alpha)$ is a local trivialisation for every $\alpha \in \Lambda$.

Theorem 4.8

Let $\mathbb{I} \in \mathrm{GL}(m, \mathbb{R})$ denote the identity matrix. Let M be a manifold with open cover $\{U_\alpha\}_{\alpha\in\Lambda}$ and family $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(m, \mathbb{R})\}$ of smooth maps satisfying[a]

1. $g_{\alpha\alpha} = \mathbb{I}$ for all $a \in U_\alpha$,
2. $g_{\alpha\beta}(a)g_{\beta\alpha}(a) = \mathbb{I}$ for all $a \in U_\alpha \cap U_\beta$,
3. $g_{\alpha\beta}(a)g_{\beta\gamma}(a)g_{\gamma\alpha}(a) = \mathbb{I}$ for all $a \in U_\alpha \cap U_\beta \cap U_\gamma$.

Then there exists a real rank m vector bundle $\pi : E \to M$ with transition functions $g_{\alpha\beta}$.

[a] these are called the **Cech cocycle conditions**.

Definition 4.9

For a vector bundle $E \to M$ we briefly denote the transition functions by $g^E_{\alpha\beta}$. Using the Cech-cocycle conditions we can define new vector bundles from old:

- Let $E \to M$ and $F \to M$ be two real vector bundles of rank k and l, respectively, over the same base space. We define the **Whitney sum** $E \oplus F \to M$ to be the vector bundle with fibres $(E \oplus F)_a = E_a \oplus F_a$ with transition functions

$$g^{E\oplus F}_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(k + l, \mathbb{R}), \qquad a \mapsto \begin{pmatrix} g^E_{\alpha\beta} & 0 \\ 0 & g^F_{\alpha\beta} \end{pmatrix}.$$

- The **dual bundle** $E^* \to M$ to $E \to M$ has fibres $(E^*)_a = \mathrm{Hom}(E_a, \mathbb{R}) = (E_a)^*$ and transition functions $g^{E^*}_{\alpha\beta} = ((g^E_{\alpha\beta})^T)^{-1}$ given by the inverse-transpose of the transition functions of $E \to M$.
- The **tensor bundle** $E \otimes F \to M$ of $E \to M$ and $F \to M$ has fibres $(E \otimes F)_a = E_a \otimes F_a$ and transition functions

$$g^{E\otimes F}_{\alpha\beta}(a) = g^E_{\alpha\beta}(a) \otimes g^F_{\alpha\beta}(a), \qquad (A, B) \mapsto A \otimes B.$$

Definition 4.17

Let $p : E \to M$ and $q : F \to N$ be two vector bundles. A pair of smooth maps (Ψ, φ) with $\Phi : E \to F$ and $\varphi : M \to N$ is a **bundle map** if for all $a \in M$ the map $\Phi_a : E_a \to F_{\varphi(a)}$ is linear with the following commutative diagram

$$\begin{array}{ccc} E & \xrightarrow{\Psi} & F \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ M & \xrightarrow[\varphi]{} & N \end{array}$$

We say that Ψ **covers** φ.

Definition 4.20: Sections

Let $p : E \to M$ be a vector bundle, then a map $s : M \to E$ is a **section** if $p \circ s = \mathrm{id}_M$, that is, if for all $a \in M$, $s(a) \in E_a$. The set of all sections of E is denoted $\Gamma(E)$.

Theorem 4.22

Let $p : E \to M$ be a vector bundle, then $\Gamma(E)$ is a $C^\infty(M)$-module.

Theorem 4.23

If $p : E \to M$ and $q : F \to M$ are vector bundles and $\Psi : E \to F$ is a bundle map (covering the identity) then Ψ defines a $C^\infty(M)$-linear map $\Psi_! : \Gamma(E) \to \Gamma(F)$ by $\Psi_!(s) = \Psi \circ s$.

Theorem 4.24

Let $E \to M$ and $F \to M$ be vector bundles with $C^\infty(M)$-linear map $\psi : \Gamma(E) \to \Gamma(F)$, then ψ is local so that if $s|_U = 0$ for $U \subseteq M$ open then $\psi(s)|_U = 0$. In particular if $s \in \Gamma(E)$ with $s(a) = 0$ then $\psi(s)(a) = 0$ as well.

Theorem 4.25

Let $E \to M$ be a vector bundle and $e \in E_a$ for some $a \in M$ then there exists a section $s \in \Gamma(E)$ such that $s(a) = e$.c

Theorem 4.26

Let $p : E \to M$ and $q : F \to M$ be vector bundles of rank k and l respectively, and let $\psi : \Gamma(E) \to \Gamma(F)$ be $C^\infty(M)$-linear. Then $\psi = \Psi_!$ for a unique bundle map $\Psi : E \to F$.

Differential Forms

Definition: Alternating Forms

A **k-linear alternating form** (or 'k-*form*') on an n-dimensional real vector space V is a k-linear map $\varphi : V^k \to \mathbb{R}$ which vanishes if any two of its arguments coincide: $\varphi(\dots, v, \dots, v, \dots) = 0$ for all $v \in V$. We will use $\bigwedge^k V^*$ to denote the set of k-linear alternating forms on V.

Theorem

Let $\varphi \in \bigwedge^k V^*$ be an alternating k-form, then:

- If φ is alternating then $\varphi(\dots, v, w, \dots) = -\varphi(\dots, w, v, \dots)$ for all $v, w \in V$ - hence the name '*alternating*', and
- $\dim\left(\bigwedge^k V^*\right) = \binom{n}{k}$.

Theorem

if $A \in \mathrm{GL}(V)$ then $(A \cdot \varphi)(v_1, \dots, v_k) = \varphi(A^{-1}v_1, \dots, A^{-1}v_k)$. This defines a Lie-Group homomorphism

$$\begin{array}{ccc} \mathrm{GL}(V) & \xrightarrow{\Psi} & \mathrm{GL}(\bigwedge^k V^*) \\ \downarrow \cong & & \downarrow \cong \\ \mathrm{GL}(n, \mathbb{R}) & & \mathrm{GL}\left(\binom{n}{k}, \mathbb{R}\right) \end{array}$$

which defines the transition functions $g_{\alpha\beta}^{\bigwedge^k E^*} = \Psi(g_{\alpha\beta}^E)$.

Definition

Taking TM and constructing $\bigwedge^k T^*M$ with transition functions $g_{\alpha\beta}^{\bigwedge^k T^*M} = \Psi(g_{\alpha\beta}^{TM})$ we get a vector bundle with the $C^\infty(M)$-module of smooth sections $\Omega^k(M)$: the **differential k-forms** on M. A typical element of $\Omega^k(M)$ is a $C^\infty(M)$-multilinear alternating map

$$\alpha : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{k-\text{times}} \to C^\infty(M), \qquad \alpha(X_1, \dots, X_k)(a) = \alpha_a((X_1)_a, \dots, (X_k)_a),$$

for all $a \in M$.

Definition

Let $F : M \to N$ be a smooth map and $\alpha \in \Omega^k(N)$, then its **pull-back** $F^*\alpha \in \Omega(M)$ by F is

$$(F^*\alpha)(X_1, \dots, X_k)(a) = \alpha_{F(a)}((F_*)_a(X_1)_a, \dots, (F_*)_a(X_k)_a).$$

Definition

Let $\alpha, \beta \in \Omega^1(M)$ then their **wedge product** (or '*exterior product*') $\alpha \wedge \beta \in \Omega^2(M)$ is

$$(\alpha \wedge \beta)(X, Y) = \det \begin{pmatrix} \alpha(X) & \alpha(Y) \\ \beta(X) & \beta(Y) \end{pmatrix} = \alpha(X)\beta(Y) - \alpha(Y)\beta(X)$$

for all $X, Y \in \mathfrak{X}(M)$.

More generally, if $\alpha_1, \dots, \alpha_k \in \Omega^1(M)$ then $\alpha_1 \wedge \cdots \wedge \alpha_k \in \Omega^k(M)$ is given by

$$\alpha_1 \wedge \cdots \wedge \alpha_k(X_1, \dots, X_k) = \det \begin{pmatrix} \alpha_1(X_1) & \dots & \alpha_1(X_k) \\ \vdots & \ddots & \vdots \\ \alpha_k(X_1) & \dots & \alpha_k(X_k) \end{pmatrix}$$

for all $X_1, \dots, X_k \in \mathfrak{X}(M)$.

Theorem 5.3

Let $\alpha, \beta \in \Omega^1(M)$ and $F : N \to M$ be a smooth map, then

$$F^*(\alpha \wedge \beta) = (F^*\alpha) \wedge (F^*\beta).$$

Definition

$$\Omega^\bullet(M) = \bigoplus_{k=0}^{n} \Omega^k(M) = \Omega^0(M) \oplus \cdots \oplus \Omega^n(M).$$

Theorem

Recall that if $(U, x^1, \dots, x^n)$ is a local coordinate chart on M then $dx^i \in \Omega^1(U)$ and every $\alpha \in \Omega^1(U)$ can be written in the form

$$\alpha = \sum_{i=1}^{n} \alpha_i X^i \quad \in C^\infty(M), \qquad X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}.$$

Definition

We say that $I = (i_1, \dots, i_k)$ is a **multi-index** of length $|I| = k$ if $1 \leq i_1 < \cdots < i_k \leq n$. We define $dx^I \in \Omega^{|I|}(U) = \Omega^k(U)$ by $dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, so that every k-form can be written as

$$\sum_{|I|=k} \alpha_I dx^I, \qquad \alpha_I = \alpha\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right).$$

Theorem 5.5

For all $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ we have

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha \quad \in \Omega^{k+1}(M).$$

Theorem 5.6

Let $F:\mathbb{R}^m\to U$ be a smooth map, then for all $\alpha\in\Omega^k(U)$ and $\beta\in\Omega^l(U)$ we have

$$F^*(\alpha\wedge\beta)=F^*\alpha\wedge F^*\beta\quad\in\Omega^{k+1}(\mathbb{R}^m).$$

Definition 5.7

The **exterior derivative** $d:\Omega^k(U)\to\Omega^{k+1}(U)$ is defined by

$$d\alpha=\sum_{|I|=k}d\alpha_I\wedge dx^I\qquad\text{where}\qquad\alpha=\sum_{|I|=k}\alpha_I dx^I.$$

Theorem 5.8

Let $\alpha\in\Omega^k(U)$ and $\beta\in\Omega^l(U)$ then

$$d(\alpha\wedge\beta)=d\alpha\wedge\beta+(-1)^k\alpha\wedge d\beta.$$

Theorem 5.9

Let $f\in C^\infty(U)$ and $df\in\Omega^1(U)$ then $d(df)=d^2f=0$. In general, if $\alpha\in\Omega^k(U)$ then $d^2\alpha=0$.

Theorem 5.11: Differential Graded Algebra

Let $F:\mathbb{R}^m\to U$ be smooth, then for all $\alpha\in\Omega^k(U)$

$$dF^*\alpha=F^*d\alpha.$$

This makes $\left(\Omega^\bullet(M),\wedge,d\right)$ into what we call a **differential graded algebra**.

Theorem 5.12: Gluing Lemma

Let $\{U_\alpha\}_{\alpha\in\Lambda}$ be an open cover for M and let $F_\alpha:U_\alpha\to N$ be smooth maps with $F_\alpha(a)=F_\beta(a)$ for all $a\in U_\alpha\cap U_\beta$. Then there exists a unique[a] smooth map $F:M\to N$ such that $F(a)=F_\alpha(a)$ for all $a\in U_\alpha$.

[a] This uniqueness allows us to 'glue' maps together, for instance if sections $s_\alpha(a)=s_\beta(a)$ for all $a\in U_\alpha\cap U_\beta$ then there is a unique global section s with $s|_{U_\alpha}=s_\alpha$.

Theorem 5.13

Let M be an m-dimensional manifold, and N be an n-dimensional manifold. Then $(\Omega^\bullet,\wedge,d)$ is a differential graded algebra and if $F:N\to M$ is a smooth map then $F^*:\Omega^\bullet(M)\to\Omega^\bullet(N)$ is a dga-morphism.

Definition

We defined the Lie derivative for tangent bundles, now for the more general definition. Let $X \in \mathfrak{X}(M)$ have local flow ψ_t so that for all $f \in C^\infty(M)$ we have the smooth function $X(f) = \frac{d}{dt}(f \circ \psi_t)\Big|_{t=0} = \frac{d}{dt}(\psi_t^* f)\Big|_{t=0}$. If $\alpha \in \Omega^k(M)$ then its **Lie derivative** along X is

$$\mathcal{L}_X \alpha = \frac{d}{dt}(\psi_t^* \alpha)\Big|_{t=0} \quad \in \Omega^k(M).$$

Definition 5.15

An $\mathbb{R}$-linear map $D : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is a **degree-k derivation** if $D : \Omega^p(M) \to \Omega^{p+k}(M)$ and D obeys the Leibniz rule

$$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^{kp} \alpha \wedge D\beta.$$

We denote the set of degree k derivations by $\mathrm{Der}_k(M)$.

Theorem 5.16

The degree k derivations form a $C^\infty(M)$-module.

Theorem 5.18

For all $X \in \mathfrak{X}(M)$ we have $\mathcal{L}_X \in \mathrm{Der}_0(M)$ and $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$.

Theorem 5.19

Let $D_1 \in \mathrm{Der}_k(M)$ and $D_2 \in \mathrm{Der}_l(M)$, then

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{kl} D_2 \circ D_1 \quad \in \mathrm{Der}_{k+1}(M).$$

Theorem 5.20

Let D be a derivation and $\alpha \in \Omega^p(M)$ and $\alpha|_U = 0$ for some open subset $U \subseteq M$. Then $D\alpha|_U = 0$.

Theorem 5.21

Let $D \in \mathrm{Der}_k(M)$ be such that $Df = 0$ and $D\alpha = 0$ for all $f \in \Omega^0(M)$ and $\alpha \in \Omega^1(M)$. Then $D = 0$.

Theorem 5.22

Let $D \in \mathrm{Der}_k(M)$ and $D \circ d = (-1)^k d \circ D$. If $Df = 0$ for all $f \in \Omega^0(M)$ then $D = 0$.

Definition 5.23

The **contraction with** $X \in \mathfrak{X}(M)$ is the $C^\infty(M)$-linear map

$$(\iota_X \alpha)(X_2, \dots, X_k) = \alpha(X, X_2, \dots, X_k).$$

Theorem 5.24

For $X \in \mathfrak{X}(M)$ we have $\iota_X \in \mathrm{Der}_{-1}(M)$ and

- $\iota_x \circ \iota_X = 0$, and
- $\iota_{fX} = f\iota_X$ for all $f \in C^\infty(M)$.

Theorem 5.25

For all $X, Y \in \mathfrak{X}(M)$ and $\alpha \in \Omega^\bullet(M)$ the following hold:

- $\mathcal{L}_X = [d, \iota_X] \in \mathrm{Der}_0(M)$, i.e. $\mathcal{L}_X \alpha = \iota_X d\alpha + d\iota_X \alpha$
- $[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]} \in \mathrm{Der}_{-1}(M)$, i.e. $\mathcal{L}_X \iota_Y \alpha = \iota_Y \mathcal{L}_X \alpha + \iota_{[X,Y]}\alpha$ and
- $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]} \in \mathrm{Der}_0(M)$, i.e. $\mathcal{L}_X \mathcal{L}_Y \alpha = \mathcal{L}_Y \mathcal{L}_X \alpha + \mathcal{L}_{[X,Y]}\alpha$.

Definition 5.27: Closed and Exact Forms

Let $\alpha \in \Omega^k(M)$. We say that α is **closed** of $d\alpha = 0$ and we say that it is **exact** if $\alpha = d\beta$ for some $\beta \in \Omega^{k-1}(M)$.

We denote the vector subspace of closed forms by $Z^k(M) \subseteq \Omega^k(M)$ and the subspace of exact forms by $B^k(M) \subseteq Z^k(M) \subseteq \Omega^k(M)$. We have the identites

$$Z^k(M) = \ker(d) \qquad \text{and} \qquad B^k(M) = \mathrm{im}(d).$$

Definition 5.28

The k**-th de-Rahm cohomology** of M is the quotient vector space

$$H^k_{dR}(M)]\frac{Z^k(M)}{B^k(M)},$$

a typical element being the equivalence of a closed form $[\alpha] = [\alpha + d\beta]$.

Theorem 5.29

- If α, β are closed then so is $\alpha \wedge \beta$,
- If α is closed and β is exact then $\alpha \wedge \beta$ is exact.

and hence the cup/wedge product $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$ is well-defined.

Theorem 5.30

If M is a connected manifold then $H^0_{dR}(M) \cong \mathbb{R}$.

Theorem 5.31

A smooth map $F : N \to M$ defines a ring homomorphism $F^* : H^\bullet_{dR}(M) \to H^\bullet_{dR}(N)$ by $F^*[\alpha] = [F^*\alpha]$. Hence $F^*([\alpha] \wedge [\beta]) = F^*[\alpha] \wedge F^*[\beta]$.

Theorem 5.32: Homotopy Invariance

Let $F : M \times [0,1] \to N$ be smooth and let $F_t(a) = F(a,t)$. Then $F_t^* : H^k_{dR}(N) \to H^k_{dR}(M)$ gives that $F_0^* = F_1^*$ for all k.

Theorem 5.33: Poincaré Lemma

Let $n > 0$ be an integer, then

$$H^k_{dR}(\mathbb{R}^n) \cong \begin{cases} \mathbb{R} & \text{if } k = 0 \\ 0 & \text{if } k > 0. \end{cases}$$

Hence $H^k_{dR}(M \times R) \cong H^k_{dR}(M)$ for any manifold M.

Integration

Definition 6.1: Partitions of Unity

A **partition of unity** on a manifold M is a collection of smooth functions $\{\rho_i\}_{i\in I}$ such that

- $\rho_i(a) \geq 0$ for all $a \in M$ and $i \in I$,
- The set of supports $\{\text{supp}\rho_i\}_{i\in I}$ is locally finite: each $a \in M$ has a neighbourhood U which intersects only finitely many of the $\{\text{supp}\rho_i\}$, i.e. $\#\{i \in I : U \cap \text{supp}\rho_i \neq \emptyset\} < \infty$.
- for all $a \in M$ we have $\sum_{i\in I} \rho_i(a) = 1$.

We know that the sum in (3) converges since (2) gives that it is a finite sum.

Theorem 6.2: Existence of Partitions of Unity

Given an open covering $\mathcal{U} = \{V_\alpha\}_{\alpha\in\Lambda}$ of a manifold M, there exists a partition of unity $\{\rho_i\}_{i\in I}$ **subordinate** to $\mathcal{U}$, that is: each $\text{supp}\rho_i \subset V_\alpha$ for some $\alpha(i)$.

Definition 6.5: Orientability

We say that a manifold M is orientable if any of the equivalent criteria hold

- $\bigwedge^n T^*M \to M$ is a trivial bundle,
- there exists a nowhere vanishing $\mu \in \Omega^n(M)$,
- M has an atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ with $\det D(\varphi_\alpha \circ \varphi_\beta^{-1}) >)$ for all $\alpha, \beta \in \Lambda$.

Definition 6.9: Orientation

An **orientation** on an orientable manifold M is an equivalence class of nowhere vanishing n-forms, where $\mu_1 \sim \mu_2$ if and only if $\mu_1 = f\mu_2$ for some nowhere zero $f \in C^\infty(M)$.

Theorem 6.10

Let $U, V \subset \mathbb{R}^n$ and $F : U \to V$ be orientation-preserving diffeomorphism with $\mu \in \Omega_c^n(V)$ a compactly-supported n-form on V. Then

$$\int_U F^*\mu = \int_{V=F(U)} \mu.$$

Theorem 6.11: Integration

Let M be an n-dimensional oriented manifold, then there exists a unique linear map (called the **integral**)

$$\int_M : \Omega_c^n(M) \to \mathbb{R},$$

so that if (U, φ) is an oriented chart and $\omega \in \Omega_c^n(U)$ then

$$\int_M \omega = \int_{\varphi(U)} (\varphi^{-1})^*\omega.$$

Theorem 6.14: Stokes' Theorem (version 1)

Let M be an oriented n-dimensional manifold and let $\omega \in \Omega_c^{n-1}(M)$ then

$$\int_M d\omega = 0.$$

Theorem 6.15

Let M be a compact, orientable, n-dimensional manifold. Then $H^n_{dR}(M) \neq 0$.

Definition 6.16: Manifold with Boundary

A set M is an n-dimensional **manifold with boundary** if it has a collection $\{U_\alpha\}_{\alpha\in\Lambda}$ of subsets and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n_+$ such that

- $\bigcup_{\alpha\in\Lambda} U_\alpha = M$
- $\varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha)$ is a bijection and $\varphi_\alpha(U_\alpha \cap U_\beta)$ is open for all $\alpha, \beta \in \Lambda$, and
- $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is the restriction of a smooth map from a neighbourhood $\varphi_\alpha(U_\alpha \cap U_\beta) \subseteq \mathbb{R}^n_+ \subset \mathbb{R}^n$ to $\mathbb{R}^n$.

The **boundary** ∂M of M is the $(n-1)$-dimensional sub manifold

$$\partial M = \bigcup_{\alpha\in\Lambda} \varphi_\alpha(\partial\mathbb{R}^n_+).$$

Theorem 6.18: Boundary Orientation

If M is an oriented manifold with boundary then there is an induced orientation on ∂M.

Theorem 6.19: Stokes' Theorem (version 2)

Let M be an n-dimensional oriented manifold with boundary ∂M and let $\omega \in \Omega_c^{n-1}(M)$ have compact support. Then, with the induced orientation

$$\int_M d\omega = \int_{\partial M} \omega.$$