Geometry of General Relativity

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These are my notes for the "*Geometry of General Relativity*" course at Edinburgh University given by José Figueroa-O'Farrill in 2019. There will undoubtedly be many mistakes in this formula sheet, and it is far from finished: given the time and energy I would add proofs of theorems, answers to exercises and workshops, and various insights. If you notice any mistakes please email me at william.bevington@zoho.eu.

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Affine Connections

Definition 1: Affine Connection

An **affine connection** on a manifold M is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad \nabla(X,Y) := \nabla_X Y$$

such that for all $X, Y \in \mathfrak{X}(M)$ and $f \in C^\infty(M)$:

- $\nabla_{fX} Y = f \nabla_X Y$,
- $\nabla_X(fY) = X(f)Y + f\nabla_X Y$.

Theorem 1

An affine connection ∇ on a chart (U, x^i) of an n-dimensional manifold M is uniquely determined by n^3 **connection coefficients** Γ^k_{ij} by the following equation

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_{k=1}^n \Gamma^k_{ij} \frac{\partial}{\partial x^k}.$$

In fact, if $X = X^i \partial_i$ and $Y = Y^j \partial_j$ then

$$\nabla_X Y = X^i \big(\partial_i Y^j \partial_j + Y^j \Gamma^k_{ij} \partial_k \big).$$

Theorem 2

Suppose ∇, ∇' are two affine connections on a manifold M, then

$$\kappa : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \qquad (X,Y) \mapsto \nabla_X Y - \nabla'_X Y$$

defines a $(1,2)$-tensor field, i.e. a section of the bundle $T^*M \otimes TM^{\otimes 2}$.

Defintion 2

The **standard affine connection** on $\mathbb{R}^n$ is given by the directional derivative, equivalently by connection coefficients $\Gamma^k_{ij} = 0$, or that $\nabla_{\partial_i} \partial_j = 0 \implies \nabla_{\partial_i}(Y^j \partial_j) = \partial_i Y^j \partial_j$.

Defintion 3

A vector-field $Y \mathfrak{X}(M)$ is said to be **parallel** to an affine connection ∇ if, for all $X \in \mathfrak{X}(M) : \nabla_X Y = 0$.

Theorem 3

Let ∇, ∇' be two affine connections on a manifold M and $\rho, \rho' \in C^\infty(M)$ with $\rho + \rho' = 1$, then

$$\rho \nabla + \rho' \nabla'$$

is an affine connection.

Theorem 4

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an atlas for M with partition of unity $\{\rho_i\}_{i \in I}$ subordinate to M, and let $\overset{\alpha(i)}{\nabla}$ be the standard affine connection for U_α. Then

$$\nabla_X Y := \sum_{i=1}^n \sum_{i \in I} \rho_i \overset{\alpha(i)}{\nabla}_X Y$$

is an affine connection on M.

Definition 4: Torsion and Curvature

Let M be a manifold with affine connection ∇, the **torsion** of ∇ is

$$T^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

and the **curvature** of ∇ is given by

$$R^\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathrm{End}(\mathfrak{X}(M)), \qquad R^\nabla(X,Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Notice that $T^\nabla(X,Y) = -T^\nabla(Y,X)$ and $R^\nabla(X,Y) = -R^\nabla(Y,X)$, and that T^∇ is a $(1,2)$-tensor field whereas R^∇ is a $(1,3)$-tensor field.

Defintion 5

We call an affine connection ∇ **flat** if the curvature $R^\nabla = 0$ and **torsion-free** if $T^\nabla = 0$.

Theorem 5

Let (U, x^i) be local coordinates for a manifold M with affine connection ∇. Then T^∇ is determined by **torsion coefficients**

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}$$

and R^∇ is determined by the **curvature coefficients**

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^m_{jk} \Gamma^l_{im} - \Gamma^m_{ik} \Gamma^l_{jm}.$$

Theorem 6: Bianchi Identities

Let ∇ be an affine connection on a manifold M with torsion T and curvature R, then

$$\mathfrak{S} R(X,Y)Z = \mathfrak{S} \nabla_X T(Y,Z) + \mathfrak{S} T(T(X,Y),Z)$$

and

$$\mathfrak{S}((\nabla_Z R)(X,Y))W + \mathfrak{S} R(T(X,Y),Z)W = 0$$

for all $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{S} A(X,Y,Z) = A(X,Y,Z) + A(Z,X,Y) + A(Y,Z,X)$.

Koszul Connections

Definition 1: Koszul Connection

A **Koszul connection** on a vector bundle $\pi : E \to M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E), \qquad (X, s) \mapsto \nabla_X s$$

such that for all $C^\infty(M)$, $s \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$:

- $\nabla_{fX} s = f \nabla_X s$, and
- $\nabla_X (fs) = X(f)s + f \nabla_X s$.

Theorem 1

Let ∇^E and ∇^F be Koszul connections on vector bundles $\pi_E : E \to M$ and $\pi_F : F \to M$, then

- $\nabla_X^{E \otimes F}(e \otimes f) = (\nabla_X^F e) \otimes f + e \otimes (\nabla_X^E f)$ is a Koszul connection on $E \otimes F$, and
- $(\nabla_X^{\mathrm{Hom}(E,F)} \varphi)(e) = \nabla_X^F \varphi(e) - \varphi(\nabla_X^E e)$ is a Koszul connection on $\mathrm{Hom}(E, F)$

for all $X \in \mathfrak{X}(M)$, $e \in \Gamma(E)$, $f \in \Gamma(f)$, $\varphi \in \Gamma(\mathrm{Hom}(E, F))$.

These constructions clearly extend a Koszul connection on $TM \to M$ to a connection on

$$T_s^r(M) := TM^{\otimes r} \otimes T^*M^{\otimes s} = \mathrm{Hom}(TM^{\otimes s}, TM^{\otimes r}) \to M.$$

Definition 2: Curvature of Koszul Connections

The **Curvature** of a Koszul connection ∇ on $\pi : E \to M$ is given by

$$R^\nabla(X, Y)s = [\nabla_X, \nabla_Y]s - \nabla_{[X,Y]} s.$$

Defintion 3

Let ∇ be a Koszul connection on the bundle $\pi : E \to M$, and let $(U, x^i, \dots, x^n)$ be local coordinates of M with local trivialisation of E given by linearly independent sections $\sigma_i, \dots, \sigma_k$. The **connection coefficients** $\omega_{ia}^b \in C^\infty(U)$ of ∇ are determined by the equation

$$\nabla_{\partial_i} \sigma_a = \omega_{ia}^b \sigma_b.$$

Parallel Transports

Definition 1: Smooth Curve

A **smooth curve** on a manifold M is a smooth map $\gamma : [0,1] \to M$. If γ_1, γ_2 are two smooth curves then their **concatenation** is

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & t \in \left[0, \frac{1}{2}\right] \\ \gamma_1(2t-1) & t \in \left[\frac{1}{2}, 1\right] \end{cases},$$

which is continuous, but *not* necessarily smooth.

Definition 2: Sections along Curves

Let $\gamma : [0,1] \to M$ be a smooth curve on a manifold M, and let $\pi : E \to M$ be a rank m vector-bundle. A **section along** γ is a smooth map $\sigma : [0,1] \to E$ such that $\pi \circ \sigma = \gamma$, that is, $\sigma(t) \in E_{\gamma(t)}$.

For example, if $s \in \Gamma(E)$ is a section then the restriction $\sigma = s \circ \gamma$ is a section along γ. If such an s exists for some section σ along γ we call σ **extendible**. Denote by[a] $\Gamma_\gamma(E)$ the vector-space of sections along $\gamma : [0,1] \to M$.

[a] Actually, $\Gamma_\gamma(E)$ is the pullback bundle $\gamma^* E \to [0,1]$ with total space $\gamma^* E = \{(t,e) \in [0,1] \times E \;:\; \gamma(t) = \pi(e)\}$.

Definition 3: Covariant Derivative

Let ∇ be a Koszul connection on a vector-bundle $\pi : E \to M$, the **covariant derivative** on $\Gamma_\gamma(E)$ **induced by** ∇ is the $\mathbb{R}$-linear map

$$\frac{D}{dt} : \Gamma_\gamma(E) \to \Gamma_\gamma(E)$$

such that

1. for all $s \in \Gamma(E)$: $\frac{D}{dt}(s \circ \gamma) = (\nabla_{\dot{\gamma}} s) \circ \gamma$ where $\dot{\gamma} = \gamma_* \left(\frac{d}{dt}\right)$, and
2. for all $f \in C^\infty([0,1])$ and $\sigma \in \Gamma_\gamma(E)$ we have

$$\frac{D}{dt}(f\sigma) = \frac{df}{dt}\sigma + f\frac{D\sigma}{dt}.$$

Definition 4: Parallel Section

We say that $\sigma \in \Gamma_\gamma(E)$ is **parallel** if $\frac{D\sigma}{dt} = 0$ or, in local coordinates,

$$\frac{d\sigma^a}{dt} + \dot{\gamma}^i(t)\omega^a_{ib}(\gamma(t))\sigma^b(t) = 0$$

for connection coefficients ω^a_{ib}.

Definition 5: Parallel Transport

Given any $v \in E_{\gamma(0)}$ there is a unique parallel $\sigma \in \Gamma_\gamma(E)$ with initial value $\sigma(0) = v$. Evaluation at $t \in [0,1]$ gives the map $\mathbb{P}_{\gamma,t} : E_{\gamma(0)} \to E_{\gamma(t)}$ given by $v \mapsto \sigma(t)$. The map $\mathbb{P}_{\gamma,t}$ is called the **parallel transport** along γ.

Theorem 1

Suppose that $s_1, s_1 \in \Gamma(E)$ are such that $\nabla_X s_1 = \nabla_X s_2 = 0$ for all $X \in \mathfrak{X}(M)$ and that $s_1(a) = s_2(a)$ for some $a \in M$, then $s_1 = s_2$.

Geodesics

Definition 1: Geodesic

Let ∇ be an affine connection on M. A smooth curve $\gamma : [0,1] \to M$ is said to be a **geodesic** for ∇ if its velocity $\dot{\gamma} = \gamma_* \left(\frac{d}{dt}\right)$, which is a vector field along γ, is *self-parallel*:

$$\frac{D}{dt}\dot{\gamma} = 0.$$

Theorem 1

Let $(U, x^1, \ldots, x^n)$ be a local coordinates on (M, ∇). We have $\gamma^i = x^i \circ \gamma$ hence $\dot{\gamma} = \dot{\gamma}^i \partial_i$. Then

$$\begin{aligned} \frac{D\dot{\gamma}}{dt} &= \frac{D}{dt}\left(\dot{\gamma}^i(t)\partial_i\right) \\ &= \ddot{\gamma}^i(t)\partial_i + \dot{\gamma}^i(t)\nabla_{\dot{\gamma}}\partial_i \\ &= \ddot{\gamma}^i(t)\partial_i + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\nabla_{\partial_j}\partial_i \\ &= \ddot{\gamma}^i(t)\partial_i + \dot{\gamma}^i(t)\dot{\gamma}^j(t)\Gamma^k_{ij}\big((\gamma(t)\big)\partial_k \end{aligned}$$

thus $\frac{D\dot{\gamma}}{dt} = 0$ if and only if

$$\ddot{\gamma}^k + \dot{\gamma}^i(t)\dot{\gamma}^j\Gamma^k_{ij} = 0. \tag{1}$$

Theorem 2

Let $a \in M$ and $v \in T_aM$ then there exists an $\varepsilon > 0$ and a unique geodesic $\gamma_{a,v} : [0, \varepsilon) \to M$ with $\gamma_{a,v}(0) = a$ and $\dot{\gamma}_{a,v}(0) = v$.

Theorem 3

Let $a \in M$, $s > 0$ and $v \in T_aM$ then there exists an $\varepsilon > 0$ and a unique geodesic $\gamma_{a,sv}(t) = \gamma_{a,v}(st)$ for all $t \in [0, \varepsilon)$.

Definition 2: Exponential Map

Let ∇ be an affine connection on a manifold $M \ni a$. The **exponential map**[a] $\exp_a : T_aM \to M$ is given by

$$\exp_a(v) = \gamma_{a,v}(1).$$

[a] So called 'exponential' because it takes us from a 'linear space' T_aM to a non-linear space M.

Definition 3: Geodesic Normal Coordinates

Let $a \in M$ for some manifold M, the **geodesic normal coordinates at** a are given by $(U, \exp_a^{-1})$ defined as follows. Let $v \in T_aM$ be such that $\exp_a$ is defined, then

$$\frac{d}{dt}(\exp_a(tv)) = \frac{d}{dt}\gamma_{a,tv}(1)\big|_{t=0} = \frac{d}{dt}\gamma_{a,v}(t)\big|_{t=0} = \dot{\gamma}_{a,v}(0) = v$$

then U is defined via the inverse function theorem as the set in which U is diffeomorphic to a neighbourhood V of $0 \in T_aM$ via $\exp_a$.

Theorem 4

In geodesic normal coordinates at $a \in M$, the connection coefficients satisfy

$$\Gamma^k_{ij}(a) + \Gamma^k_{ji}(a) = 0.$$

Theorem 5

Let $t \mapsto \gamma(t)$ be a geodesic for ∇, and $t = t(s)$ a reparametrisation (so that $\frac{dt}{ds} > 0$). The chain rule gives that $\gamma' := \frac{d\gamma}{ds} = \frac{dt}{ds}\dot{\gamma}$, then

$$\frac{D}{ds}\gamma' = \left(\frac{dt}{ds}\right)^{-1} \frac{d^2t}{ds^2}\gamma'.$$

Definition 4: Affine Parameter

An **affine parameter**[a] t for a geodesic γ is one for which

$$\frac{D\dot{\gamma}}{dt} = 0.$$

[a] Any two affine parameters s, t are related by $\frac{d^2t}{ds^2} = 0$, i.e. $t(s) = as + b$ for some constants a, b - hence 'affine'.

Riemannian Geometry

The Metric Tensor

Definition 1: Arc Length

Suppose $\gamma : [0,1] \to M$ is a smooth curve then the **arc-length** of γ is

$$\int_0^1 ||\dot{\gamma}||dt,$$

where $\dot{\gamma} = \gamma_* \left(\frac{d}{dt}\right)$.

Definition 2: Inner Product

Let V be an n-dimensional real vector space, an **inner product** on V is a non-degenerate[a] bilinear form

$$\langle -, - \rangle : V \times V \to \mathbb{R}.$$

[a]Non-degeneracy means that if $\langle \mathbf{v}, - \rangle = 0$ then $\mathbf{v} = 0$.

Theorem 1

An inner product $\langle -, - \rangle$ is determined by $\eta_{ij} = \langle e_i, e_j \rangle$ where (e_i) is a basis for V, furthermore $[\eta_{ij}]$ is a non-singular symmetric matrix.

Definition 3: Symmetric Tensor Product

The **symmetric tensor product** $\odot^2 V^* \subseteq (V^*)^{\otimes 2}$ has elements of the form $\mathbf{v}^* \otimes \mathbf{w}^* + \mathbf{w}^* \otimes \mathbf{v}^*$. We then define $\mathbf{v}^*\mathbf{w}^* = \frac{1}{2}(\mathbf{v}^* \otimes \mathbf{w}^* + \mathbf{w}^* \otimes \mathbf{v}^*)$, then

$$(\mathbf{v}^*\mathbf{w}^*)(\mathbf{v}', \mathbf{w}') = \frac{1}{2}(\mathbf{v}^*(\mathbf{v}')\mathbf{w}^*(\mathbf{w}') + \mathbf{w}^*(\mathbf{v}')\mathbf{v}^*(\mathbf{w}').$$

Then we define

$$\eta = \eta_{ij}\theta^i\theta^j$$

for canonical dual basis (θ^i) of V^* with $\theta^i(e_j) = \delta_{ij}$. This gives that $\eta(e_i, e_j) = \frac{1}{2}(\eta_{ij} + \eta_{ji}) = \eta_{ij}$ by symmetry.

Definition 4: Isometry

If (V, μ_V) and (W, μ_W) are inner product spaces (i.e. a vector space with an inner product) then a linear isomorphism

$$\varphi : V \to W$$

is an **isometry** if the pullback $\varphi^*\mu_W = \mu_V$, i.e. if for all $\mathbf{v}, \mathbf{v}' \in V$ we have $\mu_V(\mathbf{v}, \mathbf{v}') = \mu_W(\varphi(\mathbf{v}), \varphi(\mathbf{v}'))$.

Defintion 5

Let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ with inner product given by

$$[\eta_{ij}] = \begin{pmatrix} \mathbb{I}_p & 0 \\ 0 & -\mathbb{I}_q \end{pmatrix}$$

Theorem 2: Sylvesters Law of Inertia

Every n-dimensional real inner product space (V, η) is isometric to $\mathbb{R}^{p,q}$ for some fixed $p, q \in \mathbb{N}$ with $p + q = n$. We say that (V, η) has **signature** (p, q).

Definition 6: Lorentzian Vector Space

A **Lorentzian vector space** is one of signature $(n - 1, 1)$.

Definition 7: Orthonormal Basis

By Sylvester's law of inertia, every inner product space (V, η) admits an **orthonormal basis**, that is, a basis (e_i) of V such that

$$\eta(e_i, e_j) = \delta_{ij}\epsilon_i.$$

Definition 8: Metric

A **metric** of signature (p, q) with $p + q = n$ on an n-dimensional manifold M is a section $g \in \Gamma(\odot^2 T^*M)$ such that for all $a \in M$ we have that g_a defines an inner product on T_aM with signature (p, q).

In local coordinates (U, x^i) a metric has local expression $g = g_{ij}dx^i dx^j$ with $g_{ij} \in C^\infty(M)$, giving invertible matrix $[g_{ij}(a)]$. In fact $g_{ij} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x_j}\right)$.

Examples: In Euclidean space $\mathbb{R}^n$ the standard metric is given by $g = \delta_{ij}dx^i dx^j$. We define **Minkowski spacetime** to be $\mathbb{R}^n$ with metric given by $g = \eta_{\mu\nu} = dx^\mu dx^\nu$ with

$$[\eta_{\mu\nu}] = \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{I}_{n-1} \end{pmatrix}.$$

Definition 9: Riemannian Manifold

A **Riemannian manifold** is a pair (M, g) where M is a manifold and g is a metric on M. A **Lorentzian manifold** is a Riemannian manifold of signature $(n - 1, 1)$.

Theorem 3

Every manifold M admits a positive-definite Riemannian metric, and every *non-compact* manifold admits a Lorentzian metric.

Definition 10: Isometry

Let (M, g_M) and (N, g_N) be two Riemannian manifolds. A diffeomorphism $F : M \to N$ is an **isometry** if $F^*g_N = g_M$, i.e. for all $a \in M$ and $X_a, Y_a \in T_aM$:

$$g_a(X_a, Y_a) = h_{F(a)}((F_*)_a X_a, (F_*)_a Y_a).$$

We call F a **local isometry** if at $a \in M$ there is a neighbourhood $a \in U \subset M$ such that $F|_U$ is an isometry.

Definition 11: Killing Vector Field

Let (M, g) be a Riemannian manifold, a vector field $X \in \mathfrak{X}(M)$ is a **killing vector field** if

$$\mathcal{L}_X g = 0 \quad \Longleftrightarrow \quad Xg(Y, Z) = g([X, Y], Z) + g(Y, [X, Z]).$$

The Levi-Civita Connection

Theorem 1: Fundamental Theorem of Riemannian Geometry

Let (M, g) be a Riemannian manifold. There exists a unique torsion-free affine connection ∇ which is compatible with the metric; that is, $\nabla g = 0$. It is given by the **Koszul formula**

$$\begin{aligned}2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ + g(Z, [X, Y]) - g(Y, [X, Z]) - g(X, [Y, Z]).\end{aligned}$$

Definition 1: Levi-Civita Connection

The unique torsion-free affine connection compatible with metric g on a Riemannian manifold (M, g) is called the **Levi-Civita connection**. In local coordinates we have

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

where we refer to the connection coefficients Γ^k_{ij} as the **Christoffel symbols**. Denoting $g_{ij} = g(\partial_i, \partial_j)$ so that $g^{ij}g_{jk} = \delta_{ik}$ and

$$\Gamma^m_{ij} = \frac{1}{2}g^{km}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right).$$

Theorem 2

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and let X be a Killing field, then the **Killing equation** is

$$g(\nabla_Y X, Z) + g(Y, \nabla_Z X) = 0,$$

for all $Y, Z \in \mathfrak{X}(M)$. In local coordinates

$$\nabla_i X_j + \nabla_j X_i = 0$$

where $X_a = X^b g_{ba}$.

Theorem 3

Suppose (M, g) is a Riemannian manifold with Levi-Civita connection ∇ and smooth curve $\gamma : [0, 1] \to M$. Then we have $\frac{D}{dt}$ acting on vector fields along γ, $\Gamma_\gamma(TM)$, and for all $X, Y \in \Gamma_\gamma(TM)$

$$\frac{d}{dt}g(X, Y) = g\left(\frac{DX}{dt}, Y\right) + g\left(X, \frac{DY}{dt}\right).$$

Theorem 4

If γ is a geodesic of a Riemannian manifold (M, g) then $g(\dot\gamma, \dot\gamma)$ is constant along γ.

Defintion 2

Suppose (M, g) is a Lorentzian manifold (i.e. g has signature $(n-1, 1)$) and $\gamma : [0, 1] \to M$ is a geodesic along the Levi-Civita connection then we have 'causal states'

- γ is **timelike** if $g(\dot\gamma, \dot\gamma) < 0$,
- γ is **lightlike** (or '*null*') if $g(\dot\gamma, \dot\gamma) = 0$, and
- γ is **spacelike** if $g(\dot\gamma, \dot\gamma) > 0$.

The fact that geodesics are constant in a Lorentzian manifold means that the causal state of γ is fixed.

Riemann Curvature Tensor

Definition 1: Riemann Curvature Tensor

Let (M, g) be a Riemannian manifold, the **Riemann curvature tensor** is the $(0, 4)$-tensor given by

$$\mathrm{Riem} : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$
$$\mathrm{Riem}(X, Y, Z, W) = g(R(X, Y)Z, W)$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$, where the curvature is calculated via the Levi-Civita connection.

Theorem 1

The Riemann curvature tensor satisfies

$$\begin{aligned}
&\mathrm{Riem}(X, Y, Z, W) = -\mathrm{Riem}(Y, X, Z, W)\\
&\mathrm{Riem}(X, Y, Z, W) + \mathrm{Riem}(Y, Z, X, W) + \mathrm{Riem}(Z, X, Y, W) = 0\\
&\mathrm{Riem}(X, Y, Z, W) = -\mathrm{Riem}(X, Y, W, Z)\\
&\mathrm{Riem}(X, Y, Z, W) = \mathrm{Riem}(Z, W, X, Y)
\end{aligned}$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$.

Defintion 2

Let $(U, x^1, \ldots, x^n)$ be local coordinates for (M, g) then we have **Riemann coefficients** for the Riemann curvature tensor given by

$$R_{ijkl} = g(R(\partial_i, \partial_j), \partial_k, \partial_l) = g(R^m_{ijk}\partial_m, \partial_l) = R^m_{ijk} g_{ml},$$

where R^l_{ijk} are the curvature coefficients for the Levi-Civita connection.

Theorem 2

The Ricci tensor is symmetric, i.e. $\mathrm{Ric}(X, Y) = \mathrm{Ric}(Y, X)$.

Definition 3: Ricci Scalar

The **Ricci Scalar** R is given by $R := g^{ij} R_{ij}$.

Definition 4: Einstein Tensor

Let (M, g) be a Riemannian manifold. The **Einstein tensor** of its Levi-Civita connection is a symmetric $(0, 2)$ tensor field given by

$$\mathrm{Ein}(X, Y) := \mathrm{Ric}(X, Y) - \frac{1}{2} R g(X, Y).$$

It has local coordinates given by coefficients

$$G_{ij} = R_{ij} - \frac{1}{2} R g_{ij}.$$

Theorem 3

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇. The Einstein tensor is *divergence-free*

$$\nabla^j G_{jk} := g^{ij} \nabla_i G_{jk} = 0.$$

Defintion 5

A Riemannian manifold is **Einstein** if the Ricci tensor is proportional to the metric

$$\mathrm{Ric} = \lambda g \quad \text{equivalently} \quad R_{ij} = \lambda g_{ij}$$

for some $\lambda \in \mathbb{R}$. If $\lambda = 0$ we say that M is **Ricci-flat**. In fact, $\lambda = \frac{R}{n}$.

Cartan's Method of Moving Frames

Definition 1: Local Orthonormal Frame

Let (M, g) be a Riemannian manifold and let $U \subseteq M$ be open. A **local orthonormal frame** for M on U is a collection $(X_1, \ldots, X_n)$ with $X_i \in \mathfrak{X}(U)$ such that $g(X_i, X_j) = \pm\delta_{ij}$ everywhere on U.

Theorem 1

Let (M, g) be a Riemannian manifold with $a \in M$, then there exists an open neighbourhood $a \in U \subseteq M$ with a local orthonormal frame on U.

Definition 2: Connection One-Forms

Let (M, g) be a Riemannian manifold and let $(e_1, \ldots, e_n)$ be a local orthonormal frame for (M, g) on some $U \subseteq M$. The **connection one-forms** $\omega_A^B \in \Omega^1(U)$ are given by

$$\nabla_X e_A = \omega(X)_A^B e_b.$$

Theorem 2: First Structure Equation

Let (M, g) be an n-dimensional Riemannian manifold and let $(e_1, \ldots, e_n)$ be a local orthonormal frame with canonical dual co-frame $(\theta^1, \ldots, \theta^n)$ then

$$d\theta^A + \omega_B^A \wedge \theta^B = 0.$$

Theorem 3: Second Structure Equation

Let (M, g) be an n-dimensional Riemann manifold with orthonormal frame $(e_1, \ldots, e_n)$ giving canonical dual co-frame $(\theta^1, \ldots, \theta^n)$. Then

$$d\omega_B^A + \omega_C^A \wedge \omega_B^C = \Omega_B^A$$

where

$$\forall X, Y \in \mathfrak{X}(M) : \qquad \Omega(X, Y)_B^A e_A = R(X, Y) e_B$$

gives the **curvature two-form**.

Theorem 4

Let (e_A) be a local orthonormal frame with canonical dual co-frame (θ^A) on some chart (U, x^i) of a Riemannian manifold (M, g). Then we can express the Riemann curvature coefficients as

$$R_{ijkl} = -\Omega_{ijAB}\theta_k^A \theta_l^B$$

where $\Omega_{ijAB} := \Omega(\partial_i, \partial_j)_{AB}$. Equivalently, we have that

$$\frac{1}{2}\Omega_{AB}(\theta^A \wedge \theta^B) = -\frac{1}{4}R_{ijkl}(dx^i \wedge dx^j)(dx^k \wedge dx^l).$$

Theorem 5

To summarise, if we have a Riemannian manifold (M, g) then we can calculate the curvature of the metric g as follows:

1. Write $g = \eta_{AB}\theta^A\theta^B$ for some local co-frame (θ^A).
2. Solve the first structure equation for the ω^A_B subject to $\omega_{AB} = \eta_{Ak}\omega^k_B = -\omega_{BA}$.
3. Calculate the curvature two-form using the second structure equation.
4. Use theorem 4 to obtain the curvature tensor g.

Geodesic Deviation

Defintion 1

A **one parameter family of geodesics** on a Riemannian manifold (M, g) is a smooth map $\gamma : (-\varepsilon, \varepsilon) \times [0, 1] \to M$ which sends $\gamma(s, t) = \gamma_s(t)$ such that for each $s \in (-\varepsilon, \varepsilon)$ we have that $\gamma_s(t)$ is an affinely-parametrised geodesic.

Theorem 1

Let X be a vector field along Σ, i.e. $X = X^i(s, t)\partial_i$ with each $X^i \in C^\infty(M)$, then we have two covariant derivatives

$$\frac{D}{dt}X = \frac{\partial X^i}{\partial t}\partial_i + X^i \nabla_{\dot{\gamma}} \partial_i$$
$$\frac{D}{ds}X = \frac{\partial X^i}{\partial s}\partial_i + X^i \nabla_{\gamma'} \partial_i.$$

We say that γ_s is **affinely parametrised** if $\frac{D}{dt}\dot{\gamma}_s = 0$.

Definition 2: Geodesic Deviation

The **geodesic deviation** of the one parameter family of geodesics $\gamma_s(t)$ is the acceleration of γ', that is

$$\frac{D^2}{dt^2}\gamma'.$$

Theorem 2

If γ is a one parameter family of geodesics then

$$\frac{D}{ds}\dot{\gamma} = \frac{D}{dt}\gamma'.$$

Theorem 3

If γ is a one parameter family of geodesics and X is a vector field along Σ (where Σ is the surface paramatrised by $\gamma_s(t)$) then

$$\frac{D}{dt}\frac{D}{ds}X - \frac{D}{ds}\frac{D}{dt}X = R(\dot{\gamma}, \gamma')X.$$

Theorem 4: Geodesic Deviation Equation / Jacobi Equation

If γ is a one parameter family of geodesics then the deviation of γ is given by

$$\frac{D^2}{dt^2}\gamma' = R(\dot{\gamma}, \gamma')\dot{\gamma}.$$

Theorem 5

Let R be the curvature tensor of a torsion-free affine connection on a manifold M, then for all $X, Y, Z \in \mathfrak{X}(M)$ we have

$$3R(X, Y)Z = R(X, Y)Z + R(Z, Y)X - R(Y, X)Z - R(Z, X)Y.$$

Relativity

Galilean Relativity

Definition 1: Affine Space

The n-dimensional **affine space** $\mathbb{A}^n$ is given by $\mathbb{R}^n$ as a set; it is $\mathbb{R}^n$ without the origin and so we cannot add points like in $\mathbb{R}^n$. Fixing some $o \in \mathbb{A}^n$ allows us to then obtain all points in $\mathbb{A}^n$ by $o \mapsto o + \mathbf{v}$, we call $\mathbb{R}^n$ the **vector-space of parallel displacements**.

Concretely, $\mathbb{A}^n$ can be thought of as a plane $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ which doesn't go through the origin, allowing us to write $\mathbf{x} = (x_1, \ldots, x_n, 1)$ for a general point $\mathbf{x} \in \mathbb{A}^n$. A linear map between affine spaces is then given by

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} A & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 1 \end{pmatrix}$$

where $A \in \mathrm{GL}(n; \mathbb{R})$ and $b \in \mathbb{R}^n$. The **affine group** $\mathrm{Aff}(n; \mathbb{R})$ acting on $\mathbb{A}^n$ is given by any such transformation, sending $\mathbf{x} \mapsto A\mathbf{x} + b$.

Definition 2: Galilean Universe

The **Galilean universe** is given by $\mathbb{A}^4$ alongside two linear maps

- `A Clock:` $\tau : \mathbb{R}^4 \to \mathbb{R}$ where $\mathbb{R}^4$ is the vector-space of parallel displacements of $\mathbb{A}^4$. We interpret $\tau(a - b)$ as the time interval between $a, b \in \mathbb{A}^4$, and we say a and b are **simultaneous** if $\tau(a - b) = 0$, which is an equivalence relation.
- `A Ruler:` $\Delta : \ker \tau \to \mathbb{R}_{\geq 0}$, interpreted as the Euclidean distance between two simultaneous events.

Definition 3: Galilean Group

The **Galilean group** is the subgroup of $\mathrm{Aff}(4; \mathbb{R})$ preserving the clock and ruler, as a subgroup of $\mathrm{GL}(n; \mathbb{R})$ we see that the Galilean group acts by matricies of the form

$$\begin{pmatrix} R & \mathbf{v} & \mathbf{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{v}, \mathbf{a} \in \mathbb{R}^3, \quad R \in O(3), \quad s \in \mathbb{R}.$$

sending $\mathbf{x} \mapsto R\mathbf{x} + \mathbf{v}t + \mathbf{a}$.

Defintion 4

The standard clock and ruler for the Galilean universe are given by

$$\tau(t,x,y,z,1) = t$$
$$\Delta((t,x,y,z,1) - (t,x',y',z')) = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}.$$

We have the following types of basic elements of the Galilean group in this case:

- `Translation:` given by matricies of the form $\begin{pmatrix} \mathbb{I} & 0 & \mathbf{a} \\ 0 & 1 & s \\ 0 & 0 & 1 \end{pmatrix}$ sending

$$(t,x,y,z,1) \mapsto (t+s, x+a_1, y+a_2, z+a_3, 1).$$

- `Galilean Boosts:` given by matricies of the form $\begin{pmatrix} \mathbb{I} & \mathbf{v} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ sending

$$(t,x,y,z,1) \mapsto (t, x+tv_1, y+tv_2, z+tv_3, 1).$$

- `Axial Reorientation:` given by matricies of the form $\begin{pmatrix} R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ sending

$$(t,x,y,z,1) \mapsto (t, R(x,y,z)^T, 1).$$

Definition 5: World Lines

A path $\gamma : [0,1] \to \mathbb{A}^4$ traced out by a particle $\mathbf{x}$ in the Galilean universe is called the **world-line** of the particle.

Galilean transformations take inertial coordinates to inertial coordinates: if $\gamma : \mathbb{R} \to \mathbb{A}^4$ is given by $\gamma(t) = (\mathbf{x} + \mathbf{v}t, t)$ giving the world-line of $\mathbf{x}$ at constant velocity $\mathbf{v}$ then the image of γ under a Galilean transformation again takes the form $\gamma'(t) = (\mathbf{x}' + \mathbf{v}'t, t)$.

Special Relativity

Definition 6: Maxwell's Equations

Maxwell's equations are given (in Heaviside form) by

$$\nabla \cdot B = 0 \qquad \nabla \cdot E = 0$$

$$\nabla \times B = \frac{\partial E}{\partial t} \qquad \nabla \times E = -\frac{\partial B}{\partial t},$$

where $\nabla \cdot - = \partial_1 + \partial_2 + \partial_3$ and $(\nabla \times B)_i = \sum_{j,k} \epsilon_{ijk} \partial_j B_k$.

Defintion 7

Let $x^\mu = (t, x, y, z)$ be local coordinates of spacetime M and define $\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$ where

$$\eta_{\mu\nu} = \begin{pmatrix} -c^2 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \mathbb{I} \end{pmatrix}.$$

The map $x^\mu \mapsto \Lambda^\mu_\nu x^\nu + a^\mu$ preserves $\Box$ if $\Lambda^T \eta \Lambda = \eta$ which happens if and only if $\Lambda \in O(3,1) = \{\Lambda \in \mathrm{GL}(n;\mathbb{R}) : \Lambda^T \eta \Lambda = \eta\}$.

Definition 8: Minkowski Universe

The **Minkowski universe** is given by $\mathbb{A}^4$ alongside a linear map, called the **proper distance**, given by

$$\Delta(a,b) = \eta(b-a, b-a),$$

where $b - a \in \mathbb{R}^4$ and η is the Minkowski inner product so that, if $b - a = (t, x, y, z)$ then

$$\Delta(a,b) = -t^2 + x^2 + y^2 + z^2.$$

The subgroup of $\mathrm{GL}(4;\mathbb{R})$ given by transformations which preserve Δ is called the **Poincare group**.

Definition 9: Lorentz Group

The **Lorentz group** is the subgroup of the Poincare group given by affine transformations of the form

$$O(3,1)\left\{ \begin{pmatrix} \Lambda & a \\ 0 & \mathbb{I} \end{pmatrix} : \Lambda^T \eta \Lambda = \eta, \ a \in \mathbb{R}^4 \right\}.$$

Definition 10: Lorentz Boost

A **Lorentz boost** in the z-direction is a transformation of the form $(t, x, y, z) \mapsto (t', x, y, z') = (\gamma z + \delta t, x, y, \alpha z + \beta t)$ by linearity, for some $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. If we demand that this preserves the proper distance we obtain that

$$-(ct)^2 + x^2 + y^2 + z^2 = -c^2(\gamma z + \delta t)^2 + x^2 + y^2 + (\alpha z + \beta t)^2$$

so that $\alpha^2 - c^2\gamma^2 = 1$, $\delta^2 - \frac{1}{c^2}\beta^2 = 1$ and $\alpha\beta = c^2\gamma\delta$. This has solutions

$$\alpha = \cosh(\xi), \quad \gamma = \frac{1}{c}\sinh(xi), \quad \delta = \cosh(\chi), \quad \text{and} \quad \beta = c\sinh(\chi)$$

for some $\xi, \chi \in \mathbb{R}$. This gives that $\sinh(\chi - \xi) = 0$ so that $\chi = \xi$, thus a **Lorentz boost** in the z direction takes the form

$$(t, x, y, z) \mapsto \left(\frac{1}{c}z\sinh\xi + t\cosh\xi, x, y, z\cosh\xi + ct\sinh\xi\right),$$

for some $\xi \in \mathbb{R}$.

Defintion 11

A coordinate model of the Minkowski universe is **Minkowski spacetime** $(\mathbb{R}^4, \eta)$ where $\eta = \eta_{\mu\nu}dx^\mu\, dx^\nu$ with $\Gamma^\nu_{\mu\rho} = 0$. This is *flat* in the sense that geodesics satisfy $\ddot{x}^\mu = 0$ and so are straight lines. Here the Poincare group consists exactly of the isometries on $(\mathbb{R}^4, \eta)$.

Defintion 12

Fix $a \in \mathbb{R}^4$ in Minkowski spacetime. The **lightcone** at a is $L_a = \{b \in \mathbb{R}^4 \, : \, \Delta(a, b) = 0\}$ (so-called because 'light' gives a null-geodesic, so light rays starting at a lie on L_a). If $\Delta(a, b) \leq 0$ then we say that a and b are **causually related**, in particular if $\Delta(a, b) > 0$ we say that a and b are **space-like separated**.

Definition 13: Relativistic Equations

A **Relativistic equation** is a PDE on Minkowski spacetime which is invariant under the Poincare group. Maxwell's equations are relativistic.

A **Relativistic field** is a section of a vector-bundle over Minkowski spacetime (M, η), associated to which we have the **energy-momentum tensor** $T_{\mu\nu}dx^\mu\, dx^\nu$, summing two energy-momentum tensors gives another one, called the **total energy-momentum tensor** $T + T'$, which satisfies

- $T = 0$ in any open $U \subseteq M$ if and only if $T = T' = 0$ on U,
- T has zero divergence when $\eta^{\mu\nu}\partial_\mu T_{\nu\rho} = 0$.

General Relativity

Postulate 1

Spacetime is a connected four-dimensional Lorentzian manifold (M, g), which we henceforth refer to as 'the' **spacetime** (it follows that M is Hausdorff).

Postulate 2

Free particles in the spacetime follow causal (time-like or null; $g(\dot{\gamma}, \dot{\gamma}) \leq 0$) geodesics of the Levi-Civita connection.

Remark 1

The laws of physics in spacetime are governed by two principles:

- **the principle of general covariance**: the laws of physics are independent of the choice of local coordinates (coordinates have no *intrinsic* meaning), and so all equations should be of the form 'tensor= 0', and
- **the equivalence principle** that the laws of general relativity should reduce to special relativity in an inertial frame (i.e. that inertial mass is the actual mass). The laws of physics are as in Minkowski spacetime relative to geodesic normal coordinates.

This gives a simple rule, called **minimal coupling** to 'covariant-ise' Minkowski physics: in any equation simply replace the Minkowski metric η by the chosen spacetime metric g, and replace partial derivatives ∂_i by a covariant derivative ∇_i.

Postulate 3

The distribution of all matter (including radiation, but not gravity) in the spacetime is described by the energy-momentum tensor T, a symmetric $(0, 2)$-tensor field with zero divergence $\nabla^\mu T_{\mu\nu} = 0$ if the field equations are satisfied.

Note: the field equations are chosen relative to the type of matter, so we may choose to use Maxwell's equations, Klein-Gordan equations, etc.

Postulate 4

The curvature of the spacetime is related to the energy-momentum tensor via the **Einstein equation**

$$\mathrm{Ein} = 8\pi G T,$$

or, relative to a chart,

$$R_{\mu,\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu},$$

where G is Newton's gravitational constant ($G = 1$ for us).

Exact Solutions

Definition 1: Group Action

Let G be a Lie group and M a manifold. A **left action** of G on M, written $G \curvearrowright M$, is a smooth map

$$\alpha : G \times M \to M, \qquad (g, a) \mapsto g \cdot a$$

such that for all $a \in M$ and $g_1, g_2 \in G$ we have

$$e \cdot a = a \qquad \text{and} \qquad g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a.$$

Defintion 2

An action $G \curvearrowright M$ gives a homomorphism $\varphi : G \to \mathrm{Diff}(M)$ given by $g \mapsto \psi_g$ where $\psi_g(a) = g \cdot a$. We say that an action $G \curvearrowright M$ is **effective** if the corresponding map ψ has trivial kernel $\ker \psi = \{e\}$, and an action is **transitive** if for all $a, b \in M$ there exists a $h \in G$ such that $b = h \cdot a$.

Definition 3: Orbits

Let $G \curvearrowright M$, the **orbit** of some $a \in M$ is

$$G \cdot a = \{h \cdot a \,:\, h \in G\}.$$

Thus an action is transitive if and only if $G \cdot a = M$ for all $a \in M$.

Theorem 1

If G is compact then $G \cdot a$ is an embedded sub-manifold.

Theorem 2

Suppose $G \curvearrowright M$ and let $\mathfrak{g} = T_e G$ be the tangent Lie algebra. Given $X \in \mathfrak{g}$ we can construct a vector-field $\widetilde{X} \in \mathfrak{X}(M)$ called the **fundamental vector field of** X as follows.

Let $\gamma : [-\varepsilon, \varepsilon] \to G$ be a path with $\gamma(0) = e$ and $\gamma'(0) = X$ then we define a curve $c : (-\varepsilon, \varepsilon) \to M$ on M via $c(t) = \gamma(-t) \cdot a$. Then we have that $c(0) = a$ and we let $\widetilde{X}_a = c'(0)$ so that

$$(\widetilde{X} f)(a) = \frac{d}{dt} f(c(t))\Big|_{t=0}.$$

In other words, the local flow of $\widetilde{X}$ is given by $\psi_{\gamma(-t)}$ where ψ is the corresponding representation of the action as in definition 2. It follows that

$$[\widetilde{X}, \widetilde{Y}] = \widetilde{[X, Y]}.$$

Definition 4: Stabilisers

Let $G \curvearrowright M$, the **stabiliser** of an element $a \in M$ is given by

$$G_a = \{h \in G \,:\, h \cdot a = a\}.$$

Theorem 3

Let G be a Lie group with $H \leq G$ a closed subgroup, then there is a right-action $G \curvearrowleft H$ given by $(g, h) \mapsto gh$ for $g \in G$ and $h \in H$. The H-orbit of g gives the **left-coset**

$$gH = \{gh : h \in H\}.$$

Then $G/H = \{gH : g \in G\}$ is a smooth manifold such that $\pi : G \to G/H$ given by $\pi(g) = gH$ is a smooth submersive surjection.

Theorem 4: Orbit Stabiliser

Let G act smoothly on M and let $a \in M$, then there is a G-equivariant diffeomorphism

$$\varphi : G \cdot a \xrightarrow{\sim} G/G_a, \qquad g \cdot a \mapsto gG_a.$$

Definition 5: Isometries

Let (M, g) be a Riemannian manifold and $G \curvearrowright M$, we say that this action is an **isometry** if

$$\forall h \in G: \ \psi_h^* g = g,$$

where $\psi : G \to \mathrm{Diff}(M)$ is the map induced by the action, as in definition 2. This is equivalent to requiring that

$$g_a(v, w) = g_{h \cdot a}((\psi_h)_* v, (\psi_h)_* w).$$

Homogeneous Riemannian Manifolds

Definition 6: Homogeneous

A Riemannian manifold (M, g) is called **homogeneous** if it admits a transitive isometric action of a Lie group G, i.e. for all $a \in G$ we have $G \cdot a = M$ where this action preserves the metric.

Definition 7: Linear Isotropy Representation

Let $G \curvearrowright M$ for some Riemannian manifold (M, g) and set $H = G_0 = \{h \in G : h \cdot 0 = 0\}$. The map sending $h \in H$ to $(\varphi_h)_*$ is a Lie group homomorphism $H \to \mathrm{GL}(T_0 M)$ called the **linear isotropy representation** of H.

We say that the linear isotropy representation sends H to $O(g_0)$ which is the orthogonal group of g_0, which is the set of IPs on $T_0 M$.

Theorem 5: Frobenius Reciprocity

Let $G \curvearrowright M$ for some Riemannian manifold (M, g) and set $H = G_0$, then there is a one-to-one correspondence between H-invariant tensors on $T_a M$ and G-invariant tensor-fields on M. This is called *Frobenius Reciprocity*, or the *Holomony principle*.

Spherical Symmetry

Definition 8: Spherical Symmetry

Let (M, g) be a four-dimensional Lorentzian manifold on which $G = SO_3\mathbb{R}$ acts isometrically. We say that (M, g) is **spherically symmetric** if the generic G-orbits are two-dimensional spheres (by 'generic' we mean to compare the example of $M = \mathbb{E}^3$ in which for all $r > 0$ we have a sphere, but $r = 0$ gives a point).

Theorem 6

Let $M_0 = \{a \in M : G \cdot a \cong S^2\} \subset M$, then we have a smooth projection $\pi : M_0 \to M_0/G$ in which for any coordinate chart U of M we have $\pi^{-1}(U) \cong U \times S^2$. Furthermore g is non-degenerate on the orbits $G \cdot a$ for any $a \in M$ and g takes the form

$$g|_{\pi^{-1}(U)} = H(u, v) + r(u, v)^2 g_{S^2}$$

where $H(u, v)$ is the pull-back by π of a Lorentzian metric on U.

Theorem 7

here exists some function $\tau(r, s)$ such that (τ, r) are local coordinates relative to which there are no 'cross-terms' in the metric:

$$g = g_{\tau\tau}(z, t) d\tau^2 + g_{rr}(\tau, r) dr^2 + r^2 g_{S^2}.$$

The Schwarzschild Metric

Definition 1: Stationary and Static Spacetime

We say that a spacetime (M, g) is **stationary** if it admits a timelike killing vector $X \in \mathfrak{X}(M)$. A stationary spacetime (M, g) is said to be **static** if the dual one-form $\theta \in \Omega^1(M)$ to X satisfies $\theta \wedge d\theta = 0$. Such a killing vector is called **hypersurface orthogonal**.

Definition 2: Schwarzschild Metric

The **Schwarzschild metric** is given by

$$g = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2\, d\theta + r^2 \sin^2\theta\, d\varphi^2,$$

which describes a point-mass in an asymptotically flat static spacetime from an external perspective.

Definition 3: Schwarzschild Radius

The **Schwarzschild radius** is given by $r_S = \frac{2GM}{c^2}$, it is the value of r for which the Schwarzschild metric is singular. In geometric unite (i.e. $G = c = 1$) we have $r_S = 2M$.

Theorem 1

he **Kretschmann scaler** $K := \frac{1}{4} R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma}$ for the Schwarzschild metric is given by

$$K = \frac{12M^2}{r^6},$$

thus $r = 0$ is a 'physical singularity' (i.e. not an artifact of the choice of coordinates).

The Schwarzschild Black Hole

We consider the non-spherical part of the Schwarzschild metric $g = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1-\frac{2M}{r}} dr^2$ where $r > 2M$ and $t \in \mathbb{R}$. Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a null-geodesic so that $g(\dot{\gamma}, \dot{\gamma}) = 0$ which gives that

$$-\left(1 - \frac{2M}{r}\right) \dot{t}^2 + \frac{1}{1 - \frac{2M}{r}} \dot{r}^2 = 0$$

where $\dot{r} = \frac{\partial r}{\partial \lambda}$ and $\dot{t} = \frac{\partial t}{\partial \lambda}$ for some affinely parametrised λ to be determined. This gives that

$$\left(\frac{dt}{dr}\right)^2 = \left(\frac{r}{r - 2M}\right)^2.$$

We use the **Tortoise coordinates** $\hat{r}$ so that $\frac{d\hat{r}}{dr} = \frac{1}{1-\frac{2M}{r}}$ giving $\hat{r} = r + 2M \log(r - 2M) \in \mathbb{R}$.

We now use **retarded** and **advanced** coordinates $u = t - \hat{r}$ and $v = t + \hat{r}$ so that the ingoing/outgoing null geodesics satisfy $t = \pm\hat{r} + c$ for some constant c. Then $du = dt - d\hat{r} = dt - \frac{1}{1-\frac{2M}{r}} dr$ and $dv = dt + \frac{1}{1-\frac{2M}{r}} dr$ so that

$$du\, dv = dt^2 - \left(\frac{1}{1 - \frac{2M}{r}}\right)^2 dr^2 = \frac{1}{1 - \frac{2M}{r}} \left(\left(-1 - \frac{2M}{r}\right) dt^2 + \frac{1}{1 - \frac{2M}{r}} dr^2\right)$$

therefore $g = -\frac{1}{1-\frac{2M}{r}} du\, dv$ where $r(u, v)$ obeys $r + 2M \log(r - 2M) = \frac{1}{2}(v - u)$.

Dividing by $2M$ and exponentiating gives that $e^{\frac{r}{2M}}(r - 2M) = e^{\frac{1}{4M}(v-u)}$ and so we have that

$$g = -\frac{1}{r} e^{-\frac{r}{2M}} e^{\frac{v-u}{4M}} du\, dv,$$

with $u, v \in \mathbb{R}$. This is singular at $r = 0$ but **not** at the Schwarzschild radius. We define $U = -e^{-\frac{u}{4M}}$ and $V = e^{\frac{v}{4M}}$ so that $g = -\frac{16M^2}{r} e^{-\frac{r}{2M}} dU\, dV$ which is defined for $U < 0$ and $V > 0$ but we can extend this to any U, V for which $r > 0$.

Now we set $X = \frac{1}{2}(V - U)$ and $T = \frac{1}{2}(U + V)$ which gives that $-dU\, dV = dX^2 - dT^2$ so that $g = \frac{16M^2}{r} e^{-\frac{r}{2M}} (dX^2 - dT^2)$ where $r(X, T)$ satisfies $e^{\frac{r}{2M}}(r - 2M) = X^2 - T^2$. The requirement that $r > 0$ is the same as requiring that $X^2 - T^2 > -2M$ and $T^2 - X^2 < 2M$.

Robertson-Walker Metrics

"What does General Relativity say about the universe?"

Definition 1: Isotropic Space

Recall that we say that a Riemannian manifold (Σ, h) is homogeneous if there is a smooth, transitive isometric Lie group action $G \curvearrowright \Sigma$ so that, by the Orbit-Stabiliser theorem, we have that $\Sigma \cong G/H$ for some closed $H \subseteq G$ thought of as the stabiliser of some point, usually the origin. If H is the stabiliser of $o \in \Sigma$, written $H = G_o$, then there is a linear isotropic representation $H \to O(T_o\Sigma, h_o)$ given by the orthogonal group $O(T_o\Sigma, h_o)$.

We say that (Σ, h) is **isotropic** if H acts transitively on the unit sphere under this linear isotropic representation.

Theorem 1

If (Σ, h) is a strictly Riemannian (i.e h is positive-definite) homogeneous, isotropic three-dimensional manifold then (Σ, h) has constant sectional curvature

$$R(X, Y)Z = \frac{R}{6}(h(Y, Z)X - h(X, Z)Y).$$

If $R > 0$ then we say that (Σ, h) is **spherical**, if $R = 0$ we say (Σ, h) is **Euclidean**, and if $R < 0$ we say (Σ, h) is **hyperbolic**.

Theorem 2

If (Σ, h) is a strictly Riemannian (i.e h is positive-definite) homogeneous, isotropic three-dimensional manifold then we have that there exists a chart (r, θ, φ) such that

$$d = dr^2 + f(r)^2 g_{S^2}$$

where $g_{S^2} = d\theta^2 + \sin^2\theta \, d\varphi^2$ and

$$f(r) = \begin{cases} \sqrt{\frac{6}{R}} \sin\left(r\sqrt{\frac{R}{6}}\right) & \text{if } R > 0 \\ r & \text{if } R = 0 \\ \sqrt{-\frac{6}{R}} \sinh\left(r\sqrt{-\frac{R}{6}}\right) & \text{if } R < 0. \end{cases}$$

If we let $\rho = f(r)$ then for $k = R/6$ we have that

$$h = \frac{d\rho^2}{1 - k\rho^2} + \rho^2 g_{S^2}$$

Definition 2: Robertson-Walker Metric

Let (M, g) be a spacetime with spacial universe (Σ, h) which is a strictly Riemannian homogeneous, isotropic three-dimensional manifold so that $M = \mathbb{R} \times \Sigma$. In an **epoch** (an interval of the cosmological time t) where the type of spatial universe (sign of R) doesn't change then (M, g) is described by a **Robertson-Walker metric**

$$g = -dt^2 + a(t)^2 \left(\frac{d\rho^2}{1 - k\rho^2} + \rho^2 g_{S^2} \right),$$

for some function $a(t)$ and $k = 0, \pm 1$.

Theorem 3

The most general Energy-Momentum tensor compatible with homogeneity and isotropy has the components

$$T_{\mu\nu} = (p(t) + \rho(t))V_\mu V_\nu + p(t)g_{\mu\nu},$$

where $V = \frac{\partial}{\partial t}$ and p, ρ are functions of t.